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DYNAMICS

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AN INTRODUCTION

TO

DYNAMICS

INCLUDING

KINEMATICS, KINETICS, AND STATICS

WITH NUMEROUS EXAMPLES

BY CHARLES V. BURTON, D.Sc.

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TO

GEORGE HARTLEY BRYAN, M.A.

FELLOW OF ST PETER'S COLLEGE, CAMBRIDGE

THIS BOOK

IS AFFECTIONATELY DEDICATED

BY

THE AUTHOR

PREFACE

THE subjects treated of in the following pages are usually included under the head of Mechanics, the term Dynamics being reserved for the study of motion, and Statics for the study of equilibrium ; but some modern writers have pointed out that the whole Science should more properly be called Dynamics, and this nomenclature has accordingly been adopted.

In writing a book for young students with no previous knowledge of Dynamics, it was found difficult to make a satisfactory division of the subject into Kinematics, Kinetics, and Statics ; for, on the one hand, it seemed essential to examine the nature of forces before discussing their equilibrium, and to deduce the laws for their composition as far as possible from physical considerations ; while, on the other hand, many portions of Kinetics could be more satisfactorily treated after establishing some properties of the mass-centre, and these, it appeared, would be better understood when the centre of gravity had been defined.

Chapters I, II, IV, V, VI, and VII are devoted to Kinematics, and Chapter III to the trigonometry of one angle, while the tenth, eleventh, twelfth and thirteenth chapters treat chiefly of Statics, and the remaining chapters of Kinetics.

Absolute systems of units have been used almost exclusively, and the C.G.S. System has by far the most prominent place. For this, I trust, no apology will be needed.

As beginners are often greatly confused by the use of a single term in two or more distinct senses, I have tried as far as possible to avoid this source of trouble, though greater clumsiness or inelegance of phrase has sometimes resulted. As examples, the words ‘pressure’ and ‘tension’ may be mentioned; they are used to denote sometimes force per unit area, sometimes quantities of the nature of forces, and though this book is not concerned with the measurement of pressures or tensions, these terms have not been used to denote such forces as the reaction of a resisting surface or the pull of a stretched string, but have been reserved for their more legitimate use.

The few small innovations which have been introduced are intended merely as aids to the beginner. Thus, the sign ‘$(=)$’ has been used as an abbreviation for ‘is proportional to’ or ‘is represented by,’ because it appeared dangerous in an elementary work to connect by an equality two quantities of different kinds, such as forces and finite straight lines. Corresponding meanings are attached to ‘$(>)$’ and ‘$(<)$.’ Again, in chapters XV and XVI the conception without the name of vector addition has been introduced, ‘A ✢ B’ being simply an abbreviation for ‘the resultant of A and B,’ and ‘$A \leftrightarrow B$’ for ‘the resultant of A and — B.’ I hope that something may thus be gained in generality, without the risk of confusion between scalar and vector addition, and without introducing the student to any new doctrine.

Great care has been taken to secure accuracy in the text and examples, but my efforts must, I fear, have been very far from uniformly successful, and I shall be very grateful for any corrections or suggestions which may reach me. At the end of the book are some ‘additional examples,’ taken chiefly, without modification, from London University examination papers.

C. V. BURTON.

24 Wimpole Street, W.

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Errata

Page 23, Example 12, *omit.*
" 80, " 10, *for* 4 seconds *read* 1 second
" 153, " 16, *omit.*

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Errata

Page 10, lines 20 and 23, *for* 41 *read* 51
,, 295, line 18, *for* -1 cm. *read* $-\frac{14}{15}$ ft., *and for* -38 cm. *read* -36 ft.

precise, and to adhere to them very strictly, so that the definite facts and conceptions of our subject may be recorded in language which is always consistent and free from ambiguity. And when we are dealing with terms which have already become familiar to us in a more or less loose and general sense, we must be all the more careful in framing new definitions to limit their meaning, and to which (in scientific matters at least) we pledge ourselves henceforward to adhere. It cannot be too strongly urged upon the student that confusion of terms must mean for him confusion of ideas; and no pains will be spared in the following pages to avoid setting him a bad example by the use of ambiguous or inaccurate language.

2. **Measurement.**—Suppose that we are given two magnitudes of the same kind; for example, two lengths, A and B. When we can directly compare A with B, we may find, perhaps, that A is the greater. With a little more care we may find that four times the length of A is pretty nearly equal to five times

the length of B. This, then, would furnish a rough measurement of the relation between the lengths A and B: A being approximately equal to $\frac{5}{4}$ B, and B being approximately equal to $\frac{4}{5}$ A. We may, however, proceed in a less direct manner. Suppose we have a short rod of length C, and we find (as nearly as whole numbers will go) that A = 128 times C, and B = 99 times C; we shall adopt the fraction $\frac{128}{99}$ as representing more nearly than $\frac{5}{4}$ the ratio of the length A to the length B.

This second method will generally prove more convenient than the first, but we may go further. Suppose that we prepare a bar of some very permanent material, and that in all future measurements of length we agree to take the length of this bar as a standard. Accurate copies may then be made and divided into tenths, hundredths, and so on; and every observer possessing such a copy will be able to express any accessible length in terms of the 'standard' or 'unit' of length: the measurements made by different observers being rendered strictly comparable with one another.

3. **Derived Units.**—Every quantity must be measured in terms of a unit of its own kind; we cannot measure time in feet, or length in seconds. But the choice of every new kind of unit will not be altogether arbitrary. Thus, when the unit of length has been chosen, the unit of area is defined as the square with unit side, and the unit of volume as the cube with unit edge. In the same way, when the units of length and time are fixed, we may *define* that a body which moves through the unit of length in the unit of time at a uniform rate has the unit of velocity. It will be seen further on that when *three* independent units have been arbitrarily chosen, all other units may be defined by reference to these, which are accordingly called *fundamental units*, the others being called *derived units*; but for the present we shall deal with only two of the fundamental units—those, namely, of length and of time.

4. **The C.G.S. System.**—In all scientific measurements but one system of units is employed—the C.G.S., or *centimetre-gram-second system*. It dates from the year 1799, when, by order of the French National Assembly, Borda, Delambre, and

Méchain made an elaborate determination of the length of the earth's quadrant from pole to equator. A standard bar of platinum was then prepared by Borda, its length at the temperature of melting ice being $\frac{1}{10000000}$ of the quadrant. This standard is called the **mètre** (or, in English, **metre**), and is divided into 10 **decimetres**, a decimetre being divided into 10 **centimetres**, and a centimetre into 10 **millimetres**. Thus: 1 **metre** = 10 **decimetres** = 100 **centimetres** = 1,000 **millimetres.** A length of 1,000 metres is called a **kilometre**. The metre is equal to about 39·37 inches, or rather more than a yard; the centimetre being, therefore, about ·3937 inch, or not quite $\frac{2}{5}$ of an inch.

The unit of time is the **mean solar second**, or $\frac{1}{86400}$ of the mean solar day.

The definition of the **gram** is deferred to a later chapter.

The British units of length and of time are the **foot** and the **second** respectively, the remaining fundamental unit being the **pound**; and though these are probably most familiar to the student, we shall prefer to use the C.G.S. system—in the first place, because all multiples and submultiples of the centimetre and gram which are used in measurement proceed by powers of 10 (numerical calculation being thus greatly simplified); but chiefly because the C.G.S. system is exclusively used by scientific workers in every country.

CHAPTER I

VELOCITY AND ACCELERATION

5. **Kinematics** is the science of motion, apart from any conception of matter or force. It deals only with those relations between displacement, time, velocity, and acceleration which can be established by geometrical reasoning ; it is not concerned with the physical causes of motion.

6. **Displacement** is change of position. In order to specify a displacement completely, we must know both its direction and its magnitude ; thus, five feet towards the north and twelve centimetres upwards are definite displacements. A little reflection will show that displacement is only a relative term. For example, lay down a pencil on the book before you. Pull the book and pencil towards you, say, for two inches. Then push the pencil two inches away from you without moving the book. The book has now been displaced two inches relatively to the table ; and the pencil has been displaced two inches in the opposite direction relatively to the book ; while, on the whole, the pencil has received no displacement relative to the table. If, while travelling in a railway carriage, you change your seat, you receive a displacement relative to the carriage ; and this may occupy more of your attention than the far greater displacements of the train relative to the earth or of the earth in its rotation and orbital motion, in all of which you participate.

7. All the displacements which we shall at first consider are such as involve *no rotation* ; a body being said to have no movement of rotation when in each fixed direction it always presents the same aspect. Thus, a pencil which falls so that its length is always horizontal, its point always towards the east, and the name of its manufacturer always upwards, is

approximately without rotation, and its motion is therefore said to be one of **pure translation.** The earth has both kinds of movement; a *movement of translation* in its orbit round the sun, and a *movement of rotation* about its own axis.

8. **Uniform Velocity.**—Before defining how velocity is measured, we must consider what is meant by uniform velocity. If during a certain day a man walks at a uniform speed, then in each hour during that day he will have travelled the same distance; but this fact alone will not be sufficient to assure us that his velocity is uniform, for he may perhaps have rested for five minutes in each hour; and, in any case, if we examine his progress more closely, we shall find that it is not quite uniform, but proceeds more or less by a series of jerks. Or again, take the case of the minute hand of a clock whose pendulum beats seconds. In each hour the minute hand will have gone once round, in each minute it will have gone $\frac{1}{60}$ of the way round, and in each second it will have moved $\frac{1}{3600}$ of the way round. But the displacement occurring in, say, $\frac{1}{10}$ of a second will vary, and may even be nothing if the interval considered happens to fall between two ticks. Accordingly, we must define uniform velocity as follows:—

A body is said to have **uniform velocity** *during a given period when, throughout that period, it describes always equal distances in equal intervals of time, no matter how small those intervals of time may be.*

9. **Space described.**—The relation between the uniform velocity of a body, the time of motion, and the space described is one with which we are familiar. If a train is moving uniformly at the rate of v miles per hour, we know that in one hour it will have travelled v miles, and that in 10 hours it will have travelled $10v$ miles; and, more generally, that in t hours it will have travelled tv miles. For scientific purposes the unit of velocity is one centimetre per second. We are now prepared for the following definition: **Velocity** *is* **rate** *of change of position, and is measured, when uniform, by the number of units of length described in the unit of time.* We have further seen that if the uniform velocity of a body $= v$, the space s described in the time t is given by

$$s = vt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

This relation may also be written :

$$v = \frac{s}{t} \quad \text{.........................} \quad (2)$$

or

$$t = \frac{s}{v} \quad \text{.........................} \quad (3)$$

It is easy to give a direct interpretation to (2) and (3). If we know how many miles we have walked and the time we have taken to do it, we divide the number of miles (s) by the number of hours (t) to find our speed in miles per hour. If we wish to know how long the train will take between two given stations, we may divide the number of miles (s) between the stations by the speed of the train (v miles per hour). The quotient (t) will be the number of hours required.

Examples.

[NOTE.—It is usual to employ the following abbreviations : m. for metre(s) ; cm. for centimetre(s) ; mm. for millimetre(s) ; kilom. for kilometre(s) ; kilo. or kilogr. for kilogram(s) ; mgrm. for milligram(s).]

(1) A point moves with uniform velocity for one second, and during this time it travels 2,000 cm. ; find its velocity in kilometres per hour.

That is, find how many kilometres would be described in one hour by a point which moved all the while with an equal velocity. In one second the distance described is 2,000 cm. ; therefore, in 3,600 seconds the distance would be $3{,}600 \times 2{,}000$ cm. $= 72$ kilom. And thus the velocity of the point during the one second of its actual motion was 72 kilometres per hour.

(2) A carriage is moving uniformly at the rate of a mile in 10 minutes ; through what distance will it have moved in $\frac{1}{8}$ of a second ?

(3) Of two uniformly moving points, one describes m centimetres in n seconds, the other p centimetres in q seconds. Compare their velocities.

(4) Two uniformly moving bodies have such velocities that when they are moving in opposite directions the distance between them increases at the rate of 40 feet per second, and when they are moving in the same direction the distance between them increases at the rate of 10 feet per second. What are their respective velocities ?

10. **Change of Units.**—Uniform velocity being measured by

$$\frac{\text{space described}}{\text{time of motion}},$$

it will be indifferent whether we measure the space described in an hour, or a second, or a fraction of a second; for the space described will always be proportional to the time, and the value of the fraction s/t will remain unaltered. The method of changing from one set of units to another in measuring velocity will be understood from the following examples.

(i) *The measure of a velocity is* 88 *when the foot and the second are units; what will be the measure of the* **same velocity** *when the mile and the hour are units?*

The question amounts to this: How many miles per hour constitute the same velocity as 88 feet per second?

88 feet are described per second

$\therefore 88 \times 60 \times 60$ " " " " hour

$\therefore \dfrac{88 \times 60 \times 60}{3 \times 1760}$ miles " " " "

i.e., 60 " " " " "

(ii) More generally: *If the measure of a velocity is* v_1 *when the unit of length is* L_1 *and the unit of time* T_1, *what will be the measure of the same velocity when the units are* L_2 *and* T_2?

v_1 times the length L_1 is described in the time T_1

$\therefore v_1 \cdot \dfrac{T_2}{T_1}$ " " " T_2

but $$v_1 \cdot \frac{T_2}{T_1} \text{ times the length } L_1$$

$$= v_1 \cdot \frac{T_2}{T_1} \cdot \frac{L_1}{L_2} \text{ times the length } L_2$$

$\therefore v_1 \cdot \dfrac{T_2}{T_1} \cdot \dfrac{L_1}{L_2}$ times the length L_2 is described in the time T_2

$$i.e., v_1 \cdot \frac{T_2}{T_1} \cdot \frac{L_1}{L_2}$$

of the new units of length are described in the new unit of time.

11. **Variable Velocity.**—When a velocity is not uniform it

is said to be variable, and the method of measuring variable velocity requires some explanation. If we see a train passing through a station, we may judge that its speed is about thirty miles an hour; but this does not imply that we have observed the train for an hour. It may pull up in the next few hundred yards, and stand still on the rails for half the day, in which case the distance actually described in one hour will not be a correct measure of its speed while passing the station. Suppose we wish to measure the velocity of the train at a given instant, which, for convenience, may be called the instant A. We may measure the distance described in the next second, and may find perhaps that it is 44 feet. But if we take 44 feet per second (= 30 miles an hour) as measuring the speed *at the instant A*, we are *assuming* that during the whole of the second considered the speed has not varied, but has remained exactly the same as at the instant A. This assumption may or may not be correct; but probably the velocity of the train will not have altered much in one second; so that we shall not be so completely out of our reckoning as might have been the case had we measured the distance described in the next hour. If greater accuracy is required, let us measure the distance described in the $\frac{1}{1000}$ of a second immediately following the instant A. This will probably be something *very near* $\frac{44}{1000}$ of a foot; suppose we find it to be ·0441 of a foot. Then we shall feel certain that 44·1 feet per second ($= 30\frac{3}{44}$ miles per hour) *almost exactly* represents the true velocity at the instant A, for during $\frac{1}{1000}$ of a second, the velocity can hardly have changed perceptibly at all: that is, throughout the whole time considered ($\frac{1}{1000}$ second) the velocity has been almost exactly equal to the velocity at the instant A. We may now see how to measure the value of a variable velocity at any given instant:

Choose an interval of time which includes the given instant, and so short that during the whole of the interval the velocity has not time to change perceptibly; the extremely small distance described, divided by the extremely small interval of time, will measure the velocity at the given instant.

In scientific measurements the interval of time will be ex-

pressed as a fraction (t) of a second, and the distance described as a fraction (s) of a centimetre; s/t will then give the required velocity in centimetres per second.

Examples.

(1) Find approximately the velocity of a train at a given instant, it being given that during the following second six revolutions were made by the driving-wheel, three metres in circumference.

(2) A bullet leaves the muzzle of a gun with a velocity of 1,000 feet per second. In what time (approximately) will it describe the next three inches?

12. **Uniform Acceleration.**—We have already said that *displacement* is change of position, and that *velocity* is *rate* of change of position. Velocity is thus essentially different from displacement, for it introduces the further consideration of time.

If the velocity of a body changes by equal amounts in equal intervals of time—no matter how small those intervals of time may be—it is said to have a uniform acceleration.

Thus, suppose that we are travelling in a train and that we have the means of knowing its velocity at each instant. If at the commencement of our observations it has a velocity of 30 miles an hour, and 5 seconds afterwards a velocity of 31 miles an hour; and if in each succeeding 5 seconds the speed increases by an additional 1 mile per hour; and, further, if by a more minute examination we should find the same uniformity of increase—no matter how rapidly our observations succeeded one another—the train would have a uniform acceleration.

In the preceding example the velocity of the train increases by

1 mile per hour in each 5 seconds;

i.e., by $\frac{1}{5}$ „ „ „ second;

i.e., by $\frac{1}{5} \times \frac{1760 \times 3}{60 \times 60}$ feet per second in each second;

i.e., by $\cdot 29\dot{3}$ feet per second in each second.

The acceleration of the train is therefore said to be $\cdot 29\dot{3}$ *feet per second per second.*

13. We have thus seen that:

Displacement is change of position ;

Velocity is **rate** of change of position, and is measured (when uniform) by **displacement ÷ time taken** ;

Acceleration is **rate** of change of velocity, and is measured (when uniform) by **velocity gained ÷ time taken.**

A body which travels uniformly one mile in one hour has a much smaller *velocity* than one which travels 10 feet in one second, although the *space described* is greater in the first case.

A body whose velocity increases in one hour by 60 miles per hour has a much smaller *acceleration* than one whose velocity increases in one second by one mile per hour, though the *change of velocity* is greater in the first case.

It is as wrong to say that acceleration is 'change of velocity' as it is to say that velocity is 'change of position.' Velocity is **rate** of change of position, acceleration **rate** of change of velocity.

Examples.

(1) At a given instant the velocity of a train is 15 kilometres per hour, and 20 seconds afterwards it is 41 kilometres per hour. Supposing the acceleration meanwhile to have been uniform, find its value in centimetres per minute per minute.

The gain in velocity during 20 seconds was $41 - 15 = 36$ kilometres per hour $= \frac{36}{60}$ kilometres per minute $= \frac{3600000}{60}$ cm. per minute.

Therefore, the velocity added in one minute (or 3 times 20 seconds) $= 3 \times \frac{3600000}{60} = 180{,}000$ cm. per minute.

And thus the acceleration required is 180,000 centimetres per minute per minute.

(2) A body starts from rest and its motion is accelerated at the rate of 1 metre per second per second. In what time will it acquire a velocity of 36 metres per hour?

(3) If the initial velocity of a point is 30 feet per second, and its (uniform) acceleration 10,800 feet per minute per minute, what will be its velocity at the end of 10 seconds?

14. The following example will further illustrate the nature of acceleration :

If the measure of an acceleration is 32 when the unit of length

is a foot and the unit of time is a second, what will be the measure of the same acceleration when the units are the mile and the hour?

A velocity of 32 feet per second is added in each second;

$\therefore$,, $\dfrac{32}{1760 \times 3}$ miles per second is added in each second;

$\therefore$,, $\dfrac{32 \times (60 \times 60)}{1760 \times 3}$ miles per hour is added in each second;

$\therefore$,, $\dfrac{32 \times (60 \times 60) \times (60 \times 60)}{1760 \times 3}$ miles per hour is added in each hour.

That is, $78545 \cdot \dot{4}\dot{5}$ miles per hour per hour is the same acceleration as 32 feet per second per second.

More generally, *if the measure of an acceleration is* f *when the unit of length is* L_1 *and the unit of time* T_1, *what will be the measure of the same acceleration when the units are* L_2 *and* T_2*?*

The velocity added in the time T_1 is measured in the first system of units by f; hence *this same velocity* (added in the time T_1) is measured in the second system by

$$f \cdot \frac{T_2}{T_1} \cdot \frac{L_1}{L_2} \cdot \quad \text{(See § 10.)}$$

Hence, with the same acceleration, the velocity added in the time T_2 is measured by

$$f \cdot \frac{T_2}{T_1} \cdot \frac{L_1}{L_2} \cdot \frac{T_2}{T_1} \cdot = f \cdot \frac{T_2^2}{T_1^2} \cdot \frac{L_1}{L_2} \cdot$$

That is, in the second system of units, the velocity added in the unit of time (T_2) is measured by $f \cdot \dfrac{T_2^2}{T_1^2} \cdot \dfrac{L_1}{L_2}$; or the acceleration measured by f in the first system is measured by $f \cdot \dfrac{T_2^2}{T_1^2} \cdot \dfrac{L_1}{L_2}$ in the second system.

Examples on Chapter I.

(1) The diameter of the earth being about 8,000 miles, find in miles per minute the velocity of objects at the equator due to the earth's rotation.

(2) What must be the unit of velocity, that a point moving uniformly through a kilometre in fifteen minutes, may have its velocity represented by the number 4,000?

(3) Express in centimetres per second per second the acceleration 10 metres per minute per minute.

(4) A body is moving with uniform acceleration, and in one minute its velocity changes from 10 kilometres per hour to 200 metres per minute; find the acceleration in centimetres per second per second.

(5) During one hour the velocity of a body increases uniformly from 15 kilometres per hour to 45 kilometres per hour; what was the velocity at the end of the first ten minutes?

(6) If the brakes of a railway train can retard its motion at the rate of 10 feet per second per second, how long will it take to 'pull up' when the velocity is initially 40 miles per hour?

(7) If f is the measure of an acceleration when a centimetre and a second are the units of length and of time, find the measure of the same acceleration when 10 seconds is the unit of time and 10 centimetres per second the unit of velocity.

(8) What will be the unit of velocity when a metre is the unit of length, and one centimetre per second per second the unit of acceleration?

CHAPTER II

RECTILINEAR MOTIONS

15. **Convention of Signs.**—We must now refer to a convention which is very useful in specifying displacements, velocities, and accelerations. Suppose there is a line of railway which runs north and south, and that we call *northwards* the *positive* direction ; then *southwards* is to be called the *negative* direction. Thus the position of any object on the line may be specified by stating its distance (with the proper sign) from a given station (A). For example, + 5 miles from A, would mean 5 miles to the north of A ; and − 7 miles from A, would mean 7 miles south of A. Similarly with regard to velocities : + 10 miles an hour will be a northward velocity ; − 15 miles an hour will be a southward velocity.

If a train is travelling northward with an increasing speed, its *acceleration* is *northward*, and therefore *positive.* If it is travelling southward with an increasing speed, its *acceleration* is *southward*, and therefore *negative.* But suppose a train is travelling northward, with a speed *decreasing* at the rate of 5 feet per second per second, what does this imply ? In each second there is a *loss* of *northward velocity* of 5 feet per second ; that is, in each second there is a *gain* of *southward velocity* of 5 feet per second ; so that the *acceleration* is *southward* (or *negative*). Similarly, if the velocity is *southward* and *decreasing* the *acceleration* is *northward* (or *positive*).

It will be obvious that, if the velocity and acceleration have the same sign, the train will be moving faster and faster ; but if the velocity and acceleration have opposite signs, the movement will be slower and slower. Of course it is quite a matter

of indifference whether we call northwards or southwards the positive direction ; but, having once decided this point, we must be consistent through the whole investigation, and measure all such quantities as displacements, velocities and accelerations according to the same scheme.

16. The truth of the following remarks will be obvious : a train standing still on the rails has neither velocity nor acceleration ; a train moving at a uniform speed has velocity, but no acceleration. A train *just* starting, or *just* stopping has acceleration (either positive or negative) but no velocity. For exercise the student should write down the answers to the following examples, and *afterwards* compare them with the answers at the end of the book.

Examples.

Write down the sign (whether +, − or 0) of the displacement, velocity and acceleration in each of the following cases. (*Northwaras* being taken as the *positive* direction.)

(1) Train north of station, and moving towards station with ncreasing speed.

(2) Train south of station, and moving towards station with decreasing speed.

(3) Train south of station, and moving away from the station with decreasing speed.

(4) Train passing through station, and travelling southwards at a uniform speed.

(5) Train arrived at station from the south, and just at the instant of stopping.

(6) Train north of station, and just at instant of starting to move away from station.

17. We will now consider the following problem :

A body starts with a velocity of 1,000 *centimetres per second and is subject to a uniform acceleration of* 200 *centimetres per second per second in the opposite direction. Find the velocity at the end of* 8 *seconds.*

Let us agree to call the original direction positive, then the acceleration will have the negative sign.

The velocity added in each second is − 200 centimetres

per second; so that the velocity added in 8 seconds is -1600 centimetres per second.

At the end of the 8 seconds, then, the velocity will be $1000-1600$ centimetres per second $= -600$ centimetres per second.

The motion of the body will thus be in the opposite direction to what it was at starting. After the first 5 seconds, the original velocity has just been destroyed, and the body is *instantaneously at rest* (that is, there is a single instant when the velocity is neither positive nor negative). In the next three seconds after this, the acceleration, still acting in the negative direction, produces a velocity of -600 centimetres per second. By adopting a simple convention with regard to positive and negative signs, we were enabled above to make the calculation in a single step.

18. **Composition of displacements along one given straight line.**—The first point to decide is, which direction along the given straight line is to be called positive; the opposite direction being, of course, called negative. Then let a point receive successively a number of displacements along the line, some in one direction, and some in the opposite direction (i.e. some positive and some negative). The *resultant* displacement of the point will be the *algebraic sum* of all the *component* displacements. Thus let the point receive successively the displacements a, b, c, d, e; and let $a = +10$ cm., $b = -7$ cm., $c = -2$ cm., $d = +3$ cm., $e = -11$ cm. The total displacement of the point from its original position will now be

$$\begin{aligned} & a+b+c+d+e \\ &= (+10-7-2+3-11) \text{ cm.} \\ &= -7 \text{ cm.}; \end{aligned}$$

or seven centimetres in the negative direction; a, b, c, d, e are called *component* displacements, and the displacement of -7 centimetres which they produce collectively, is called their *resultant*.

19. **Velocities along the same straight line** may be compounded in a precisely similar way, as may be seen from the following illustration:

A tramcar is travelling uniformly at the rate of 9 *feet per second, and a passenger walks down the middle of the car towards the conductor's end, at a uniform rate of* 4 *feet per second. What is the velocity of the passenger relative to the ground?*

Let us choose the positive direction to be that in which the tramcar is moving. The velocity of the passenger is made up of two components. A velocity of -4 feet per second, relative to the tramcar, and a velocity of $+9$ feet per second, due to the motion of the tramcar itself. In one second the displacement of the tram is $+9$ feet, and the displacement of the passenger relative to the tram -4 feet; hence the resultant displacement of the passenger (relative to the ground) $= +5$ feet; and this will be the displacement in each second, so long as the motion continues uniform; that is, the *resultant* velocity is $+5$ feet per second, which is the *algebraic sum* of the component velocities $+9$ per second and -4 feet per second. The same principle may evidently be extended to any number of component velocities *along the same straight line*, the resultant velocity being the algebraic sum of the components. We shall return to this point later on.

20. The same law will hold if the velocities are not uniform. For, consider a **very short** interval of time (t), following a given instant A (see § 11). Let the tramcar's velocity at the instant A be v_1, and the passenger's velocity relative to the car v_2. Then in the (very short) time t, the displacement of the tramcar is $v_1 t$, and the displacement of the passenger relative to the car is $v_2 t$; hence the *resultant* displacement of the passenger (relative to the ground) is $v_1 t + v_2 t = (v_1 + v_2) t$ in the very short time t. Hence $v_1 + v_2$ is the *resultant velocity* of the passenger at the instant A; $v_1 + v_2$ being, of course, the *algebraic* sum of the component velocities v_1 and v_2.

21. This principle may now be applied to find the distance described in a given time by a body whose motion is uniformly accelerated. To commence with, take a numerical example:

At a given instant (A), *a train is moving with a velocity of* $+40$ *miles per hour, and it has a uniform acceleration of* $+60$ *miles per hour per hour. Find the distance described in the* 20 *minutes following the instant A.*

In one hour, the increase of velocity would be 60 miles per hour (with the present rate of acceleration). Therefore, in each minute the velocity increases by one mile per hour, and at the end of twenty minutes from the instant A, the velocity will be $40 + 20 = 60$ miles per hour.

Suppose that on the same line of rails, a good many miles behind, another train is moving all the while with a *uniform velocity* of fifty miles per hour. At first, this uniformly moving train will be *gaining* on the other, at the rate of ten miles per hour. But this *relative* speed will gradually decrease, until at the end of ten minutes, *both* trains are moving at fifty miles an hour, and their relative velocity is zero. But the velocity of the foremost train goes on increasing at the same uniform rate as before, so that at the end of another ten minutes its velocity is sixty miles an hour, and it is therefore *outstripping* the other at the rate of ten miles per hour. The *relative motion* of the two trains during the 20 minutes may now be completely summed up as follows. During the first ten minutes *the hinder train gains* on the one in front at a rate which begins by being ten miles per hour, and diminishes uniformly to zero; during the second ten minutes *the front train gains* at a rate which begins by being zero, and increases uniformly to ten miles per hour. It will, therefore, be evident that during the second ten minutes the front train *gains* on the other, exactly as much ground as it *lost* during the first ten minutes. That is, at the end of the twenty minutes, the distance between the two trains is exactly the same as it was at the instant A; which means, of course, that each train has covered the same distance in the twenty minutes. But the hindermost train was moving all the while with a uniform velocity of fifty miles an hour.

The distance travelled by *each* train in twenty minutes is therefore,

$$\frac{20}{60} \times 50 \text{ miles}$$
$$= 16\tfrac{2}{3} \text{ miles}$$

The train whose velocity changed uniformly from 40 to 60 miles per hour during the twenty minutes, has described the same distance as if its velocity had been all the while uniform,

and equal to the arithmetic mean of 40 and 60 miles an hour. We are thus prepared for the following proposition.

22. *If a body move for a time* t, *with a velocity which changes uniformly during that time from* v_1 *to* v_2, *the space described will be the same as if the velocity had been uniform during the time t, and equal to the mean of* v_1 *and* v_2, that is :

$$s = \tfrac{1}{2}(v_1 + v_2)\,t \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

This proposition is perfectly general, but due account must be taken of the *signs* of v_1 and v_2. Thus, the mean of $+3$ centimetres per second and -5 centimetres per second is $\frac{+3-5}{2} = -1$ centimetre per second ; and the mean of the velocities $+v$ and $-v$ is $\frac{+v-v}{2} = 0$; so that, if the initial and final velocities are equal and opposite, the space described will be zero ; always supposing, of course, that the *acceleration* is *uniform*. This means that if a body starts with a velocity $+v$, and goes on under a negative acceleration until it stops and turns back again, its velocity will have become equal and opposite to the initial velocity v, just as it is passing again through the starting-point (the whole space described being then obviously zero).

Examples.

(1) A body starts from rest, and in 20 seconds acquires a velocity of 80 centimetres per second ; the change of motion having taken place uniformly. What displacement has meanwhile been described ?

Take the direction of the acceleration as positive ; then the velocity changes uniformly from zero to 80 cm. per second during 20 seconds of time, and the corresponding displacement will be

$$\tfrac{1}{2}(0 + 80) \times 20 \text{ cm.} = +800 \text{ cm.} ;$$

that is, 800 centimetres in the direction of the acceleration.

(2) A point whose initial velocity is 600 metres per second is subject to an opposite acceleration of 20 centimetres per second per second. Find the distance described during the next hour.

The acceleration being -20 cm. per second per second, the change of velocity during one hour will be $-20 \times 3{,}600$ cm. per

second $= -720$ metres per second; and the final velocity will be $600-720 = -120$ metres per second.

The displacement accomplished during the 3,600 seconds of the motion is therefore

$$\tfrac{1}{2}(600 - 120) \times 3{,}600 = +864{,}000 \text{ metres};$$

that is, 864 kilometres in the direction of the initial velocity.

(3) A uniformly accelerated point starts with a velocity of 20 feet per second, and at the end of a minute is found to have travelled 480 feet in the opposite direction. Find the value of the acceleration, and also of the final velocity.

(4) A body starts with a velocity of 30 feet per second, and in 10 seconds returns to its starting-point. Determine the acceleration, supposing it to have been uniform.

23. Some important formulæ may now be deduced.

A body starts with an initial velocity u, *and moves under a constant acceleration* f *for the time* t. *Find the velocity at the end of the time, and also the space described.*

The velocity *added* in the time t by the acceleration f is, of course, ft.

So that the final velocity $= u+ft$ (every quantity being supposed to have its proper *sign*). Also, the initial velocity being u, the *mean* of the initial and final velocities

$$\tfrac{1}{2}(u+v) = \tfrac{1}{2}(u+u+ft) = u + \tfrac{1}{2}ft \qquad (5)$$

The space which *would be* described in the time t with the *uniform* velocity $u + \frac{1}{2}ft$

$$= (u + \tfrac{1}{2}ft)t$$
$$= ut + \tfrac{1}{2}ft^2;$$

and, by § 22, this is equal to the space actually described under the given conditions. We therefore write

$$s = ut + \tfrac{1}{2}ft^2 \qquad (6)$$

It was also found above that the final velocity

$$v = u + ft \qquad (7)$$

24. As a particular case, suppose that the **initial velocity is zero**, (6) then becomes

$$s = \tfrac{1}{2}ft^2 \qquad (8)$$

which, of course, is only applicable when the motion considered is from rest, and the acceleration constant.

Examples.

(1) We may illustrate the use of equation (6) by applying it to Ex. (2) of § 22. Thus $u = 600$ metres per second, $f = -20$ cm. per second per second $= -\cdot 2$ metres per second per second, $t = 3{,}600$ seconds. Whence

$$\begin{aligned} s &= ut + \tfrac{1}{2}ft^2 \\ &= 600 \times 3{,}600 + \tfrac{1}{2}(-\cdot 2) \times (3{,}600)^2 \text{ metres} \\ &= 2{,}160{,}000 - 1{,}296{,}000 = 864{,}000 \text{ metres,} \end{aligned}$$

as before.

(2) A body starts from rest, and has a uniform acceleration of 800 centimetres per second per second. Find the space described in 10 seconds.

In equation (8) we have $f = 8$ metres per second per second, $t = 10$ seconds, and

$$s = \tfrac{1}{2}ft^2 = \tfrac{1}{2} \cdot 8 \cdot 10^2 = 400 \text{ metres.}$$

(3) A body moves from rest under a constant acceleration for 8 seconds, and thus describes 120 feet. What velocity has been acquired?

(4) A point whose motion is uniformly accelerated moves over 27 metres in the fifth second from rest. Determine the acceleration.

(5) Show that the space described in the nth second from rest under the uniform acceleration f, is equal to $\frac{2n-1}{2} \cdot f$ units of length.

25. **Falling Bodies.**—If a leaden bullet, a piece of cork, and a feather are let fall simultaneously from the same height, they will not all reach the ground at the same time. The bullet will be first, then the cork, and finally the feather. If, however, we perform the experiment in a closed vessel, from which all the air has been removed by an air-pump, the bullet, the cork, and the feather will fall exactly side by side, and will reach the bottom of the vessel together. For convenience, we may refer to this as the '*vacuum experiment.*'

It has been shown experimentally, by methods which we shall afterwards study more closely, that every body when falling freely *in vacuo* moves with a **constant acceleration** which is directed **vertically downwards**; and it has further been shown

that this acceleration is **the same for all bodies** at a given place on the earth's surface; this last relation being confirmed by the 'vacuum experiment.'

26. The acceleration of falling bodies is different, however, in different localities and at different heights above the sea-level, its value being usually denoted by g. In these islands the value of g near the sea-level is about 32·2 feet per second per second, or 981 centimetres per second per second; that is, a body freely falling from rest (*in vacuo*) will have, *at the end of the first second*, a velocity of 32·2 feet per second, or 981 centimetres per second; at the end of n seconds a velocity of $32·2 \,.\, n$ feet per second, or $981 \,.\, n$ centimetres per second.

g is **not a force**; different bodies have very different weights, that is, are attracted downwards with different forces;

And g is **not a velocity**, for the velocity of a freely falling body is continually increasing. But the velocity of a freely falling body *changes at a constant rate*, which is *the same for all bodies at the same place*;

g **is an acceleration**; it is measured in feet per second per second, or in centimetres per second per second.

The motion of bodies which are rising or falling vertically may be investigated by means of equations (6), (7), and (8) of §§ 23 and 24, the acceleration being denoted by g in place of the more general symbol f.

Examples.

[In the following examples the value of g may be taken as 980 cm. per second per second.]

(1) A stone is projected vertically upwards with a velocity of 1,470 centimetres per second; to what height will it ascend, and how long will it take to reach the highest point of its path?

The downward direction being called positive, the initial velocity $u = -1{,}470$ cm. per second, and the acceleration $g = +980$ cm. per second per second. After any time t, the velocity is

$$v = u + gt,$$

and when $v = 0$ this becomes

$$u + gt = 0, \text{ or } t = -\frac{u}{g} = -\frac{-1{,}470}{980} = +1·5 \text{ seconds};$$

so that the highest point will be reached 1·5 seconds after the instant of projection.

The space described meanwhile is

$$\tfrac{1}{2}(u + 0)t = \tfrac{1}{2}(-1{,}470) \cdot \tfrac{3}{2} = -1{,}102{\cdot}5 \text{ cm};$$

and since the downward direction has been taken as positive, this means that the body will rise to a height of 1,102·5 centimetres above the point of projection before it comes momentarily to rest. After another 1·5 seconds the body will be passing again through the point of projection with a velocity equal and opposite to u.

(2) A body is projected vertically downwards, with a velocity of 980 centimetres per second. How long will it take to return to the point of projection?

It is evident that so long as the downward acceleration (g) remains unchanged, the body will continue to descend, and will never return to its original position. There is, however, a theoretical solution which is easily interpreted.

The displacement acquired after any time t is given by

$$s = ut + \tfrac{1}{2}gt^2,$$

and the instant sought for is when the displacement from the original position is zero, in which case

$$ut + \tfrac{1}{2}gt^2 = 0, \text{ or } t(u + \tfrac{1}{2}gt) = 0.$$

There are two solutions to this equation: $t = 0$, and $u + \tfrac{1}{2}gt = 0$. The first of these corresponds to the instant of projection; the second solution gives $t = -2$ seconds, which shows that the body would have occupied the same position two seconds *previously*, supposing the uniformly accelerated condition to have dated so far back.

(3) A stone is dropped from the top of a tower 98 metres high, and another stone is simultaneously projected upwards from the foot of the tower with a velocity which would just carry it to the top; find when and where the stones will meet.

Examples on Chapter II.

In all the following examples the motion is supposed to be uniformly accelerated, and in the case of falling bodies the value of g may be taken as 980 centimetres per second per second.

(1) If a body thrown vertically upwards takes 5 seconds to return to its starting-point, find the velocity of projection.

(2) A stone is projected vertically downwards with a velocity of 7,550 centimetres per second; what is the time taken to describe the third metre?

(3) One stone is projected upwards from a point on the ground, and simultaneously another stone is dropped from a height; if they reach the ground at the same instant, compare the velocities which they have then attained.

(4) A body moves through 500 centimetres in the third second from rest; find the acceleration.

(5) A body moves through 200 metres, and meanwhile its velocity changes from 15 to 10 metres per second; how long will the body take to come to rest, and how far will it have travelled altogether?

(6) During 10 seconds a body moves over 12 metres, and its velocity is then found to be 25 metres per second; what is the value of the acceleration?

(7) If a body describes 12 metres and 10 metres in two successive seconds of its motion, when will it come to rest?

(8) Two bodies are projected vertically upwards, one three seconds before the other, and they reach simultaneously the highest points of their paths. If the first body ascends twice as high as the second, find when each will return to its starting-point.

(9) A body travels 100 metres in the nth second of its motion, and 150 metres in the $(n + 4)$th second; at what time was the body at rest?

(10) A stone is dropped from a height, and three seconds afterwards a second stone is dropped from half the height; if the two stones reach the ground together, what will then be their velocities?

(11) If, at the same place, two bodies are projected vertically at any times and with any velocities, show that their *relative* velocity will remain unaltered so long as they both move freely under the action of gravity.

(12) A body is thrown vertically downwards with a velocity of 1,760 centimetres per second, and it describes the last $\frac{9}{16}$ of its path to the ground in two seconds; at what height from the ground was the point of projection?

(13) If a body starts with a velocity of 9 metres per second, and in the first second of its motion it describes 7 metres, find the acceleration.

(14) The velocity of a body changes from 350 to 200 centimetres per second while it describes 15 metres; find the time taken.

(15) What is the measure of the acceleration of falling bodies when a metre and a minute are the units of length and of time respectively?

(16) If 98,000 be the measure of the acceleration of falling bodies when 5 seconds is the unit of time, what is then the unit of velocity?

CHAPTER III

TRIGONOMETRY OF ONE ANGLE

27. It would hardly be possible in subsequent chapters to avoid some reference to trigonometrical functions, and, accordingly, this chapter has been inserted for the sake of those students who have no previous knowledge of the subject.

28. Let N O R be a given angle, and from any points, P, Q, R, in O R, let fall the perpendiculars P L, Q M, R N, on O N; then the triangles O P L, O Q M, O R N, are equiangular to one another, and the ratio

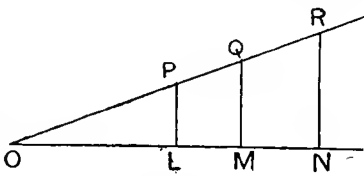


FIG. 1.

$$\frac{LP}{OP} = \frac{MQ}{OQ} = \frac{NR}{OR} \quad \text{[Euclid VI. 4.]}$$

Any of these equal ratios (L P/O P or MQ/O Q or N R/O R, etc.) is called the **sine** of the angle NOR, and it may be seen from the above that the sine of an angle is completely determined when we know the angle itself; for the ratio LP/OP has been shown to be independent of the distance of P from the vertex (O) of the angle.

29. We now proceed to define an angle, and the sine of an angle, with greater generality. The definitions will appear more natural and reasonable in the light of subsequent reading; but if any part of them should at first seem somewhat arbitrary, let us remember that all that is really requisite for a sound and useful definition is, that it shall clearly lay down what does and what does not come within the meaning of the

term defined. Provided this condition is satisfied, we may so frame our definitions as to include those quantities or relations which are most useful in practice; but, having once laid down what is meant by a term, we must be careful always to use it consistently with the definition.

Thus, we have already found it convenient, in dealing with displacements, velocities, and accelerations, to designate opposite directions by the + and − signs, and in this way we arbitrarily extended the meaning of the word *velocity*, for we do not consider two bodies to have the same velocity unless both the speed and direction of their motion is the same. But on the other hand, this arbitrary definition enabled us to represent both speed and direction along a given straight line by means of a single quantity, and led to much greater simplicity in our equations.

30. The **angle** between two straight lines, is the angle through which the first line must be rotated to make it coincide in *direction* with the second; a rotation in the direction of the hands of a clock being considered *negative*, and a rotation in the contrary direction *positive*. Thus, in fig. 1, the angle N O R, through which O N must be turned to coincide in direction with O R, is a positive angle; while the angle R O N, through which O R must be turned to coincide in direction with O N, is negative. The magnitude of an angle is usually measured in **degrees**, a degree being $\frac{1}{360}$th part of a complete rotation, or $\frac{1}{90}$th part of a right angle.

To determine the angle A O B between two given straight lines (O A, O B, fig. 2), let a circular arc, M N, be described with its centre at the vertex of the angle, commencing at a point M, in the first line O A, and terminating at N, where it meets the second line O B. If the arc M N contains $\frac{n}{360}$ of a whole circumference, then A O B is an angle of n degrees, or, as it is written for shortness, n°.

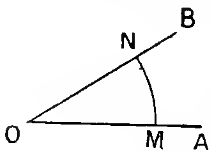


Fig. 2.

There is another unit of angular measurement called the

radian, which is mostly used in theoretical work: it is the angle determined by a circular arc whose length is equal to its radius. Thus, in fig. 2, if the length of the arc M N is equal to O M or O N, the angle A O B is an angle of one radian. Now, the length of the circumference of any circle is equal to π times its diameter, π being a factor which is the same for every circle, and roughly equal to $\frac{22}{7}$, or more nearly to 3·14159. Hence, a circumference is equal to 2π radii, and an angle of four right angles to 2π radians, one right angle being, therefore, equal to $\frac{\pi}{2}$ radians.

The number of radians contained in an angle is called its **circular measure**, and the circular measure of an angle may also be defined as the ratio,

$$\frac{\text{length of arc}}{\text{radius}};$$

the arc being described as in fig. 2. Suppose, then, that s is the length of a circular arc, θ the angle which it subtends at the centre, and r the radius of the circle. We shall have

$$\theta = \frac{s}{r}, \text{ or } s = r\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

a relation which will be subsequently useful.

From these definitions it will appear that there is no limit to the value, positive or negative, which an angle may have. Thus, in any given interval of time, the minute hand of a clock describes a certain (negative) angle, which serves to measure the interval, according to that clock; in fifteen minutes the angle is $-90°$, in one hour $-360°$, in twenty-four hours $-8{,}640°$, and so on.

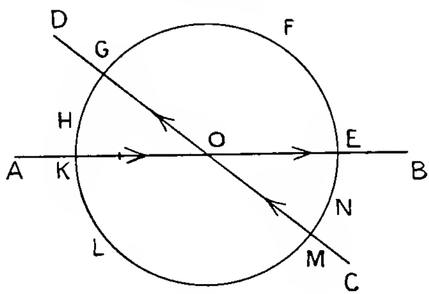


FIG. 3.

31. **Angular Distance.**—Let A O B, C O D, be two straight

lines, whose directions are indicated by the order of lettering, or by the arrow-heads (fig. 3). The angle between these lines, as laid down in § 30, is the angle through which A O B must be rotated, to make it coincide in direction with C O D ; that is, the positive obtuse angle B O D, corresponding to the arc E F G, or else the negative angle B O D, greater than two right angles, corresponding to the arc E L G. It must be observed that the acute angle B O C is *not* the angle between A O B and C O D, for if A O B were rotated through this angle, its direction would then be *opposite* to that of C O D.

The angle B O D (arc E F G or arc E L G) may be called the *angular distance of* C D *from* A B ; and, in the same way, either of these angles is the angular distance of D C from B A ; for the angle between two straight lines is not altered by reversing the directions of *both*, that is, by turning each through two right angles.

On the other hand, the angular distance of D C from A B, or of C D from B A, is the angle B O C (arc E N M or arc E F H L M) ; and it will also be evident that if R and Q be any two given directions, the angular distance of R from Q is equal and opposite to the angular distance of Q from R.

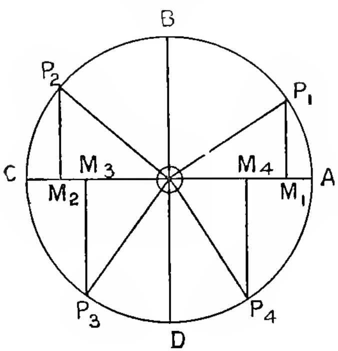


Fig. 4.

32. **Changes of Sign.**—Let O A be a straight line (fig. 4), and O P_1 another straight line, making, with the former, an angle A O P_1 which is positive and less than one right angle : and let fall the perpendicular P_1 M_1 upon O A. Then $P_1 M_1/O P_1$ is the sine of the angle A O P_1, or, as it is more shortly written,

$$\sin A O P_1 = M_1 P_1/O P_1$$

Now let A O P_2 be a positive angle, whose value lies between one and two right angles (that is, an obtuse angle). Let fall the perpendicular

$P_2 M_2$ on $A O$ produced ; then this perpendicular, divided by $O P_2$, is the sine of the angle $A O P_2$, or

$$\sin A O P_2 = M_2 P_2 / O P_2.$$

Let $A O P_3$ (arc $A B C P_3$) be a positive angle lying between two and three right angles in magnitude, and let fall the perpendicular $P_3 M_3$ on $A O$ produced ; then

$$\sin A O P_3 = M_3 P_3 / O P_3.$$

Finally, let $A O P_4$ (arc $A B C D P_4$) be a positive angle between three and four right angles, and let fall the perpendicular $P_4 M_4$ on $O A$; then

$$\sin A O P_4 = M_4 P_4 / O P_4.$$

In each case the angle we deal with is the angular distance *from* $O A$ of some second straight line ($O P_1$, $O P_2$, $O P_3$, or $O P_4$), and in each case the rule is to let fall a perpendicular on the first straight line from some point in the second, whose distance from the vertex is commonly called the radius. The ratio of the perpendicular to the radius is then called the sine of the angle.

We must now explain what *signs* are to be attributed to these various quantities. In the first place, the *radius* is always to be considered *positive*, or, to speak more generally, it is never supposed to suffer a change of sign, so that the sign to be attributed to the sine of any angle depends only on that of the perpendicular.

Two perpendiculars will be affected with the same or contrary signs, according as they lie on the same or on opposite sides of $C A$; and we further adopt the convention that the sine of an acute positive angle is positive, so that in the diagram $P_1 M_1$ and $P_2 M_2$ are positive, while $P_3 M_3$ and $P_4 M_4$ are negative.

If the circumference of the circle $A B C D$ is divided into quadrants by the diameters $C A$, $B D$, we may call $A O B$ the *first quadrant*, $B O C$ the second, $C O D$ the third, $D O A$ the fourth ; for this is the order in which the quadrants would be described by a radius originally coinciding with $O A$, and rotated in the positive direction. Accordingly, $A O P_1$ is said to

be an angle in the first quadrant, A O P_2 in the second, A O P_3 in the third, and A O P_4 in the fourth. Angles in the first and second quadrants have *positive sines* ; while angles in the third and fourth quadrants have *negative sines.*

It is not usual in practice to consider angles which are of greater magnitude than two right angles ; the angular distance of O P_3 from O A, for example, being taken as the negative angle A O P_3 (arc A D P_3), which serves equally well to determine the direction ; the *sine* of the negative angle A O P_3 being, of course, $M_3 P_3/O P_3$ as before, for, by definition, the sine of an angle depends only on the directions of the straight lines which contain it.

33. Consider now the variations which occur in the sine of an angle contained between O A and another radius O P, which at first coincides with O A, and is made to rotate positively through four right angles.

When O P coincides with O A, and, therefore, P with A, the perpendicular P M is zero, and so, therefore, is the sine of the angle A O P, or

$$\sin 0^\circ = 0^\circ.$$

As P moves along the arc A P_1 B, the length of the perpendicular increases, until when P coincides with B, and P M with B O, the perpendicular has become equal in magnitude to the radius. At the same time the angle A O P has become identical with A O B, or one right angle, so that

$$\sin 90^\circ = 1,$$

or, n circular measure,

$$\sin \frac{\pi}{2} = 1.$$

If the point P now continues its progress along the arc B P_2 C, the perpendicular P M will remain positive, but will gradually diminish in value until P coincides with C when we shall have

$$\sin 180^\circ = 0, \text{ or } \sin \pi = 0.$$

Similarly it will be evident that

$$\sin(-180^\circ) = \sin 180^\circ = 0, \text{ or } \sin(-\pi) = 0,$$

and that

$$\sin(-90^\circ) = -1, \text{ or } \sin\left(-\frac{\pi}{2}\right) = -1.$$

34. **The Cosine.**—The distance from the vertex of an angle to the foot of the 'perpendicular' is commonly called the 'base,' and the cosine of an angle is the ratio of the base to the radius. Thus, in fig. 4, the cosine of the angle AOP_1 is OM_1/OP_1; or, more shortly,

$$\cos AOP_1 = OM_1/OP_1$$

and, similarly, $\cos AOP_2 = OM_2/OP_2$, &c.

As stated before, the radius is never supposed to change its sign; while two bases will be affected with the same or contrary signs, according as they are measured from O in the same or in opposite directions. We also adopt the convention that the cosine of an acute positive angle is positive, so that OM_1 is positive, and, therefore, also OM_4, while, on the other hand, OM_2 and OM_3 are negative. There will, then, be positive cosines for angles in the first and fourth quadrants, and negative cosines for those in the second and third.

Let us now consider once more the case of a movable radius OP, which at first coincides with OA and is made to rotate about O in the positive direction (fig. 4). When P is at A, the base OP coincides with OA, and the cosine of AOP is unity; thus:

$$\cos 0^\circ = 1.$$

As P moves round the arc AB, the length of the base continually diminishes, until, when OP coincides with OB and M with O, OM has become zero; that is,

$$\cos 90^\circ = 0, \text{ or } \cos\frac{\pi}{2} = 0.$$

After the position B has been passed by the point P, the base will become negative, and will gradually increase in magnitude until P coincides with C, when OM and OC will also be coincident. This gives

$$\cos 180^\circ = -1, \text{ or } \cos\pi = -1.$$

And, similarly, it may be seen that

$$\cos(-180°) = -1, \text{ or } \cos(-\pi) = -1,$$

and that

$$\cos(-90°) = 0, \text{ or } \cos\left(-\frac{\pi}{2}\right) = 0.$$

35. The **Tangent** of an angle is defined as the ratio of the perpendicular to the base, or, with our usual notation,

$$\tan AOP = MP/OM$$

In the first quadrant the perpendicular and the base are both positive, and in the third quadrant they are both negative, so that in these two quadrants the tangent is always positive. In the second quadrant the perpendicular is positive and the base negative, and in the fourth quadrant the perpendicular is negative and the base positive, the tangent being therefore negative in both these quadrants.

The truth of the following remarks will also be evident from an inspection of fig. 4. The tangent of an angle of 0 or π is equal to zero. The tangent of an angle a *little less* than a right angle is *large* and positive; the tangent of an angle a *little greater* than a right angle is *large* and *negative*; and the tangent of a right angle is infinitely great, and may be either positive or negative: thus

$$\tan\frac{\pi}{2} = \pm\infty$$

and, similarly,

$$\tan\left(-\frac{\pi}{2}\right) = \mp\infty.$$

36. We shall now prove the following relations:

$$\left.\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \tan(-\theta) &= -\tan\theta \end{aligned}\right\}$$

Let AOP be the angle θ, and AOP' the angle $-\theta$, fig. 5. Draw the circle PAP' with O as centre, and with any radius cutting OA, OP, OP', as in the figure. Since the straight line PMP' is evidently bisected perpendicularly by OMA, $MP' = -MP$; and $OP' = OP$ (each being considered positive), $\sin(-\theta) = MP'/OP = -MP/OP = -\sin\theta$.

Again, $\cos(-\theta) = OM/OP' = OM/OP = \cos\theta$. And $\tan(-\theta) = MP'/OM = -MP/OM = -\tan\theta$.

These propositions will be true whatever be the value of θ, for if $AOP = \theta$, and $AOP' = -\theta$, P and P' will always be symmetrically situated on opposite sides of OA; MP' will always be equal to $-MP$, while OM will be the same for θ as for $-\theta$.

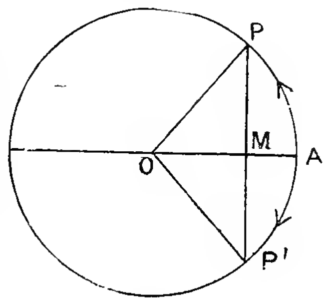


FIG. 5.

37. Since the sine of an angle is equal to the perpendicular divided by the radius, and the cosine is equal to the base divided by the radius, we have

$$\text{sine} / \text{cosine} = \text{perpendicular} / \text{base} = \text{tangent},$$

or,
$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

whatever be the value of θ.

Again, it will be seen from fig. 4 that the 'radius' is always the hypotenuse of a right angled triangle of which the perpendicular and the base are sides; and therefore, in all cases,

$$(\text{perpendicular})^2 + (\text{base})^2 = (\text{radius})^2$$

or,
$$(\text{perpendicular})^2 / (\text{radius})^2 + (\text{base})^2 / (\text{radius})^2 = 1$$

that is,
$$(\sin\theta)^2 + (\cos\theta)^2 = 1;$$

or, as it is usually written,

$$\sin^2\theta + \cos^2\theta = 1,$$

whatever be the value of θ.

38. The following table gives the values of the sine, cosine, and tangent for certain angles between 0° and 180°.

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\pm\infty$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

In practical applications we should usually have to deal with other angles than those given in the table, but in designing problems with a view to illustrating principles, we shall choose our numerical data so that tedious calculations are avoided.

Examples on Chapter III.

(1) Adopting four right angles as the unit of angular measurement, find the measure of a degree and of a radian.

(2) If an angle is positive and less than four right angles, between what limits must it lie that its tangent may be negative?

(3) The point B is north-north-east of A, and the point C is east-south-east of A. Find the angle between the directions of C A and A B.

(4) Show, geometrically or otherwise, that as the sine of angle increases in magnitude the cosine decreases, and *vice versâ*.

(5) Find the cosine and tangent of an angle of 135°.

(6) Find the sine of an angle of 120°.

(7) Show that the square of the tangent of an angle increased by unity is equal to the square of the reciprocal of the cosine.

(8) What angle has a sine equal to $-\frac{1}{2}$?

(9) What angle has a cosine equal to $-\frac{1}{\sqrt{2}}$?

(10) The square of the cosine of an angle is equal to 3 times the square of its sine; find the angle.

(11) If two angles differ by 90°, show that the sine of the one is numerically equal to the cosine of the other.

(12) If the sine of one angle is identical with the cosine of a second angle, and the cosine of the first with the sine of the second, what is the relation between the angles?

CHAPTER IV

GRAPHIC METHODS

39. IF a road joining two given stations A and B is 'straight' in the ordinary acceptation of the term, but is *not level*, we may record the configuration of its hills and valleys in either of two ways. Having fixed upon some standard of level, L (such as the sea-level), we may select a number of points P_1, P_2, &c., along the road between A and B ; and we may construct a table setting forth the horizontal distance of each point, P_1, P_2, &c., from the station A, together with its height above the level L. If P_1, P_2, . are taken sufficiently close together—say 10 feet apart—the table will give us a very fair idea of the undulations in the road between A and B. Or, in place of the table, we may construct a diagram : Suppose that a vertical section of the road is made by a plane passing through A and B ; the form of this section may be represented

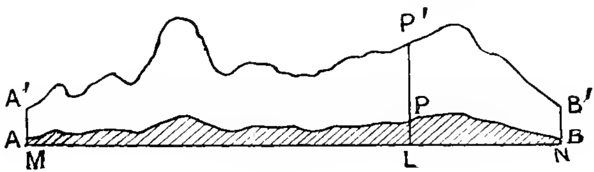


FIG. 6.

as in fig. 6, the line MN being at the standard level L, and the sinuous line between A and B which bounds the section is evidently a correct representation of the undulations of the road. Suppose the section has been drawn to a scale of 1 inch to the mile. If we want to know the height of the road above MN at a point distant M miles (horizontally) from A, we

measure M L, equal to M inches, and draw L P perpendicular to M N, meeting A B in P. If L P $= \frac{1}{n}$ of an inch, then we know that at the given point the height of the road above the level L $= \frac{1}{n}$ of a mile. It may be found that when the diagram is drawn 'to scale,' the *heights* (such as L P) cannot be made of a convenient size without making M N unmanageably long. It will then be best to represent *horizontal* distances according to one scale (say 1 inch to a mile) and vertical distances according to a larger scale (say 4 inches to the mile). The diagram A′ B′ thus produced will no longer be a *picture* of the section, but it will not on that account be any the less useful for practical purposes. We can still measure horizontal distances along M N, and the vertical measurements such as L P′ will give us the corresponding heights according to a known scale. Moreover, the diagram will still show at a glance whereabouts are the greatest elevations and depressions in the road, where the hills are steepest, and so on ; and in this respect it will be far superior to a table of numerical values.

40. Diagrams of this kind may be equally useful when the quantities concerned are not lengths at all ; for example, lengths measured horizontally may be taken to represent intervals of time. Thus, in fig. 7, if the point O corresponds to 12 (noon), and if the points M_1, M_2, M_3, &c., placed successively at equal distances along O A, correspond to 1 P.M., 2 P.M., 3 P.M., &c., each of the lengths O M_1, M_1 M_2, M_2 M_3, &c., corresponds to one hour, and if O M_1, &c., each = 1 centimetre, we are representing time along the line O A on a scale of one centimetre to the hour ; although, of course, a centimetre and an hour are quantities of totally different kinds. Such a diagram as this may be used to record the variations of the thermometer : thus, suppose we find that at 12 (noon) the temperature is

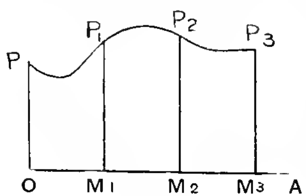


FIG. 7.

$15°$ C, and that we draw O P vertically upwards and equal to 15 millimetres; similarly, let the temperature at 1 P.M. be $18°$, at 2 P.M. $19°$, at 3 P.M. $17°$, and let us draw the vertical lines $M_1 P_1 = 18$ mm., $M_2 P_2 = 19$ mm., $M_3 P_3 = 17$ mm. We are thus representing temperatures measured from $0°$ C. by vertical heights above OA on a scale of one millimetre to the degree. If the observations of temperature are made more frequently than every hour, a number of points between P, P_1, P_2, P_3, will be obtained, and by making observations at sufficiently short intervals, we can get something like a continuous record of temperature during the time of observation. When a number of points have been obtained sufficiently close together, it is usual to complete the diagram by drawing a continuous curve as smoothly as may be, and passing through them all. Let P P_1 P_2 P_3 in the figure be such a curve: at 12 the thermometer stood at $15°$; it then fell until about 12.20, when it stood at $13°$; from this time it continued to rise until about 1.45, when the temperature was $20°$. There was now a pretty steady fall of temperature down to $17°$ at 2.25, at which point the thermometer remained nearly stationary until 3 o'clock.

41. In future we shall require to use the following terms:

Lengths measured along O A (i.e. *horizontally*) are called **abscissæ** (singular, **abscissa**). Lengths measured perpendicularly to O A (i.e. *vertically*) are called **ordinates**. Now, let us study the motion of a point along a straight line by constructing a curve whose abscissæ represent time, and whose ordinates represent velocity; and first take the case in which the velocity is uniform. If O P represent the velocity according to any convenient scale (fig. 8), then all the ordinates, O P, $M_1 P_1$, $M_2 P_2$, &c., will be equal to OP, and the 'velocity-curve' will be a straight line P Q parallel to O A.

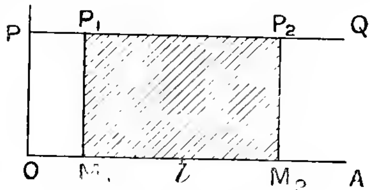


FIG. 8.

Consider, now, any interval of time, such as is represented

by $M_1 M_2$. The velocity during this time is represented by $M_1 P_1$ or $M_2 P_2$, and the space described, being the product of the time and the velocity, will be represented (according to the scale of our diagram) by the product of $M_1 M_2$ and $M_1 P_1$; that is, by the area $M_1 P_1 P_2 M_2$ included between $M_1 M_2$, the ordinates $M_1 P_1$, $M_2 P_2$, and the velocity-curve $P_1 P_2$.

42. Next in simplicity is the case of uniformly accelerated motion. Let the velocity at the instant O (=) O P,[1] and consider two successive equal intervals of time (=) $O M_1$ and $M_1 M_2$, so that $O M_1 = M_1 M_2$ (fig. 9). Let the velocities at the times M_1, M_2 (=) $M_1 P_1$, $M_2 P_2$ respectively; draw $L P_1 N$ parallel to O A, as in the figure, and join $P P_1$, $P_1 P_2$. It must not be assumed without proof that $P P_1$, $P_1 P_2$ are in the same straight line, but the proof is not difficult.

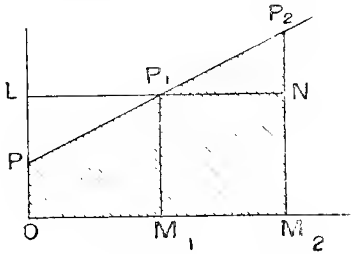


FIG. 9.

$P L = O L - O P = M_1 P_1 - O P$ (=) the increase of velocity during the time represented by $O M_1$.

Similarly, $N P_2$ (=) the increase of velocity during the time (=) $M_1 M_2$. But $O M_1 = M_1 M_2$, and the acceleration is uniform; $\therefore P L = N P_2$.

Hence, in the right-angled triangles $L P_1 P$ and $N P_1 P_2$, $L P_1 = P_1 N$ and $P L = N P_2$; therefore, the two triangles are equal and similar and the angle $L P_1 P =$ the angle $N P_1 P_2$. Therefore, P, P_1, P_2, are in one straight line.

But P, P_1, P_2 are the extremities of the ordinates O P, $M_1 P_1$, $M_2 P_2$, and are therefore three points on the velocity-curve. Similarly, if we consider any number of successive equal intervals of time (great or small), the corresponding points on the velocity-curve will all lie in one straight line.

Hence, when the acceleration is uniform, the velocity-curve

[1] We cannot accurately say that the velocity = O P, since O P is a *length* and *not* a velocity at all. The sign (=) is used in this book as an abbreviation for such expressions as 'corresponds to,' 'is represented by.'

is a straight line, and, consequently, the straight line $P\,P_1\,P_2$ is a portion of the velocity-curve in our diagram.

The acceleration, being uniform, may be found by dividing the increase of velocity (=) $N\,P_2$, by the time (=) $M_1\,M_2$, in which that increase of velocity occurred. Thus, on the scale of our diagram the (uniform) acceleration of a body whose velocity-curve is $P\,P_1\,P_2$, will be represented by $N\,P_2/P_1\,N$; that is, by *the tangent of the angle* $N\,P_1\,P_2$. The student who is unacquainted with trigonometry will have no difficulty in realising that when the scale of the diagram is fixed, the (uniform) acceleration depends only on the inclination of $P\,P_1\,P_2$ to the horizontal.

43. Suppose, now, that we are dealing always with uniformly accelerated motion; and further suppose, for simplicity, that the initial velocity in each case is that represented by O P. First, take a velocity-line, P Q, which slants upwards towards the right, and whose inclination to the horizontal is nearly a right angle. Take any abscissa, O M, and draw the ordinate M L Q as in the figure; also draw P L parallel to O A.

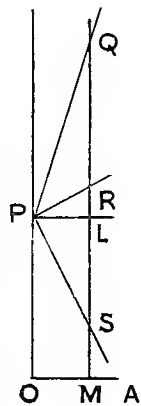


Fig. 10.

Now, the velocity added in the time (=) O M is (=) L Q, so that the acceleration is (=)L Q/P L or tan L P Q, and is consequently very great. The nearer L P Q approaches to a right angle, the greater is L Q/P L, that is, the greater is the acceleration. If the inclination of the velocity-line is somewhat smaller—say L P R—the acceleration will be correspondingly less, namely L R/P L, which is still, of course, positive. If the inclination of the velocity-line to the horizontal is zero, that is, if the line itself is horizontal (P L), the acceleration is zero. In other words, the velocity is uniform, as was also seen in § 30.

Now, take a case in which the velocity-curve, P S, slants downwards towards the right. The velocity added during the time represented by O M is (=) L S, which is negative; and the acceleration being (=) L S/P L is also negative. Observe

that LS/P L is still the tangent of the angle L P S, this angle being measured *from* the positive horizontal direction *to* the direction of the velocity-curve.

Hence, in all cases of uniformly accelerated motion, the acceleration is proportional to the tangent of the angle between the velocity-line and the horizontal, taken with its proper sign.

44. Now, take the most general case, in which the velocity varies in any manner. Let P R S T U V W Q be the velocity-curve during the time considered. At the time O, the velocity is positive and (=) O P ; the moving body continues from this time to gain speed until the instant M, when the velocity begins to decrease again. The diminution of velocity continues till the time L, and from this time there is an increase until a maximum is again reached at the time K. At U, the

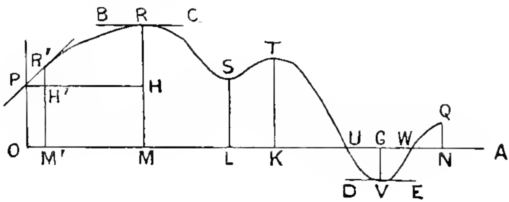


Fig. 11.

velocity has become zero, that is, the body has stopped ; the velocity, however, still continues to *decrease*, so that its value becomes negative, the motion of the body being in the opposite direction to what it was at first. At the time G, the velocity is represented by G V (which is, of course, negative), and after this it begins to *increase* again (that is, to become *numerically less*, since its value is negative), and, at the instant W, the body has once more stopped. The velocity still increasing, again becomes positive, till, at the instant N, when our diagram ceases, the body is moving in its original direction with a velocity represented by N Q.

45. **Acceleration at any Instant.**—Draw P H horizontal to meet M R in H. The velocity at the time O (=) O P, and the velocity at the time M (=) M R. Hence, in the time (=)

O M the *increase* of velocity (=) M R − O P, which = H R, and is evidently positive here. This does *not* necessarily give the *rate of increase* of velocity at the instant O, unless the acceleration is uniform. Let us consider a *very short* interval of time, (=) O M′; draw the ordinate M′ H′ R′, as in the figure. Then during an interval of time (=) O M′ the increase of velocity is (=) H′ R′, and the acceleration *at the instant* O will be *very nearly* (=) H′ R′/P H′, provided that the interval of time (=) P H′ is so small that the acceleration has not time to change to an appreciable extent.

Let us look at the matter from another point of view. If a number of points be taken very close together along the velocity-curve, and if each such point be joined to its immediate neighbours by straight lines, as in fig. 12, the resulting broken line will not differ greatly from the original smooth curve; and the nearer together the points P, P_1, P_2, &c., are taken, the less will the broken line differ from the curve. Thus, when R′ is sufficiently close to P in fig. 11, we may replace the portion of the curve P R′ by the straight line P R′, and the acceleration at the instant O depends upon the angle H′ P R′, which P R′ makes with the horizontal. Thus the acceleration at the instant O is very nearly (=) H′ R′/P H′, which = tan H′ P R′. But the straight line drawn through two very neighbouring points on the curve (P and R′) is almost exactly the tangent at P, and, when P H′ is made *indefinitely small*, we arrive at the following proposition:

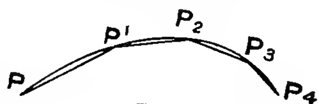


FIG. 12

The acceleration at any instant depends upon the direction of the tangent at the corresponding point of the velocity-curve, and is represented (according to a certain scale) by the tangent[1] *of the angle which this tangent*[1] *makes with the horizontal.*

46. Let us now consider how the space described can be

[1] It will be noticed that the word *tangent* is here used in two utterly different senses.

represented on such a diagram. Let C P T U Q (fig. 13) be the velocity-curve, and let M N represent a short interval of time. Draw the ordinates M P, N Q to the velocity-curve ; produce N Q to R, and draw P R, Q S horizontal as in the figure. Now, if during the entire interval of time (=) M N the velocity had been (=) M P, the space described would have been represented by the area M P R N (see § 30). Similarly, if during the whole time (=) M N the velocity had been (=) N Q, the space described would have been represented by the area M S Q N.

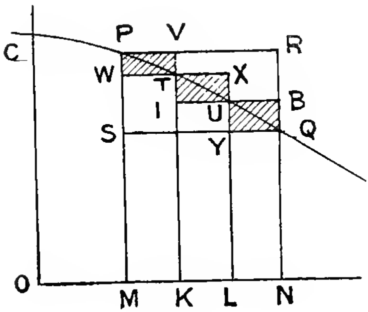


FIG. 13

But since during the time (=) M N the velocity has never been (>) M P[1] or (<) N Q, the space actually described cannot be (>) M P R N, or (<) M S Q N. We observe that the area bounded by M N, M P, N Q, and the curve P Q lies between these two limits, and we feel tempted to infer that here is the correct value of the quantity we are seeking ; a little closer inspection will show that we are right. Divide the interval M N into any number of parts M K, K L, L N, and draw the ordinates K T, L U corresponding to the points K, L. Complete the construction as in the figure. Then, employing the same mode of reasoning as before, the space described in the time (=) M K will be (=) something between the rectangles P K and W K, with similar limits in the case of the intervals K L, L N. Thus the whole space described in the time represented by M N (or M K + K L + L N) is (<) the sum of the rectangles P K + T L + U N, and (>) the sum of the rectangles W K + I L + Y N. These

[1] For (>) read 'represented by a quantity >,' or '> the quantity represented by' ; with an analogous meaning for (<).

two limits differ by the sum of the shaded rectangles, and are therefore a good deal closer together than the limits formerly assigned, which differed by the rectangle S P R Q.

We observe, then, that by dividing the interval M N, we have been able to narrow the limits within which the required area must lie; we observe, too, that the area M P T U Q N, which seemed likely to represent the correct value, still lies between these narrower limits.

As we sub-divided the interval M N, so we may further sub-divide each of the intervals M K, K L, L N ; and as the area S R was replaced by the smaller aggregate area of the shaded rectangles, so will each of these shaded rectangles give place to a still smaller area. Thus we may go on indefinitely dividing and sub-dividing, and as we do so the limits between which the true area is known to lie will continually get closer and closer together, while the area M P T U Q N will also lie between these limits. We now feel quite convinced that we were right, and that M P T U Q N *is the true value of the area which, on the scale of our diagram, represents the distance described in the time* (=) M N, *when the corresponding portion of the velocity-curve is* P T U Q.

47. On looking back at fig. 13, it will be seen that something was assumed in the foregoing investigation: it was assumed that during the whole time represented by M N the velocity was always decreasing. During any interval of time in which the velocity is always increasing or always decreasing, our argument will be strictly applicable ; but, to complete the demonstration and make it perfectly general, let us turn back for a moment to fig. 11 (p. 40).

During the time (=) O M the velocity is always increasing, so the space described is (=) O P R M ; during the time (=) M L the velocity is always decreasing, so the space described is (=) M R S L. Similarly, during the intervals (=) L K, K U the spaces described are (=) L S T K and K T U respectively. Thus, in the whole interval of time, (=) O U, the space described is (=) the area O P R S T U O. But how about the portion of the curve U V W which dips down on the other side of O A? The diagram was constructed on the under-

standing that ordinates drawn *downwards* from O A were to represent *negative* velocities, that is, velocities in the negative direction. But the space described by a body moving during any time with a negative velocity will obviously be negative; so that areas which lie *below* the line O A must be reckoned *negative*.

Thus, in the time represented by O N, the whole space described will be represented by the area O P R S T U − U V W + W Q N.

48. The graphic method is most useful practically, when the motion to be investigated is subject to irregular variations; in cases of uniform or uniformly accelerated motion, the calculations are most easily made without reference to a diagram. By way of illustration, however, we shall apply the method to prove equation (6) of § 23; namely,

$$s = ut + \tfrac{1}{2}ft^2;$$

where u is the initial velocity, f the (uniform) acceleration, and s the distance described in the time t.

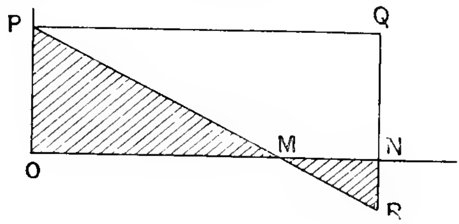


FIG. 14.

Draw O P (=) u, (fig. 14), and P Q = O N (=) t. Then, since the acceleration f is constant, the velocity-curve will be a straight line drawn from P to some point R in Q N (produced if necessary), R being such that

$$QR/PQ\ (=)\ f.$$

In the figure Q R is drawn downwards, and this corresponds to a negative value of f.

The distance described in the time t (=) the area bounded by P R, O N, and the ordinates O P, N R; such portions of

the area as lie above O N being reckoned positive, and the portions below, negative.

In the figure we shall have, s (=) the triangle P O M − the triangle M N R = the rectangle O P Q N − the triangle P Q R.

Now, Q R/P Q (=) f, and P Q (=) t; ∴ Q R (=) ft, and the area of the triangle P Q R = $\frac{1}{2}$ P Q · R Q (=) $-\frac{1}{2}t \cdot ft$ (=) $-\frac{1}{2}ft^2$.

Also, since O P (=) u, the area O P Q N (=) ut.

Hence, the area O P Q N − P Q R (=) $ut - (-\frac{1}{2}ft^2) = ut + \frac{1}{2}ft^2$; and this is consequently the space described. Similarly, equation (6) may be verified, whatever positive or negative values be given to u and to f.

Examples on Chapter IV.

(1) If the abscissæ of a diagram represent times on a scale of one centimetre to the second, and the ordinates velocities on a scale of one centimetre to one metre per second, what length will be represented by an area of one square centimetre, and what will be the acceleration of a body whose velocity-line slants downwards to the right at an angle of 45° to axes of the diagram?

(2) A body starts with a velocity of five metres per second, and its motion is retarded at the rate of two metres per second per second. Draw a diagram to illustrate this case, and hence calculate the space described in four seconds.

(3) Show, by means of a diagram, that when a body moves from rest under a uniform acceleration, the space described is proportional to the square of the time of the motion.

(4) A diagram is constructed whose abscissæ represent times on a scale of one minute to the centimetre, while the ordinates represent accelerations on a scale of 100 centimetres per second per second to the centimetre. How would you represent on such a diagram, (*a*) uniform motion, (*b*) uniformly accelerated motion?

(5) In the last question, what quantity would be represented by an area of one square centimetre, and what kind of

motion would be represented by a straight line inclined to the axes of the diagram?

(6) On a diagram whose abscissæ represent times and whose ordinates represent displacements along some given straight line, what kind of motion will be represented, (*a*) by a straight line parallel to the 'time' axis, (*b*) by a straight line inclined to the axes?

(7) A body starts with a velocity of ten centimetres per second, and during the first five seconds of its motion the acceleration is -3 centimetres per second per second; the acceleration then suddenly changes, its value during the next three seconds being $+5$ centimetres per second per second. What is now the velocity of the body and over what distance has it travelled?

CHAPTER V

COMPOSITION OF DISPLACEMENTS

49. We must now consider more particularly what is meant by displacement, velocity, and acceleration, and especially what is meant by saying that two displacements, or velocities, or accelerations are *equal* to one another.

The ordinary sense of the word velocity is simply the *magnitude* of the velocity ; and in this sense we have sometimes used the word *speed*. This is really the only meaning of the word which we need for every-day use. Thus we may reckon how long it will take us to get from one station to another if we have some notion of the distance between the two, and of the average speed of the train. But even here (perhaps unconsciously) we are taking the *direction* of motion into account ; for we should find ourselves quite out of our reckoning if we took a down train instead of an up train, or travelled on a wrong line of railway.

In dealing with movements along one straight line, two velocities have been called equal when the magnitude and direction of each were the same. Two points moving along a straight line with equal speeds in opposite directions were said to have equal and opposite velocities, and the velocity of either was said to be equal to *minus* the velocity of the other.

50. It will be found convenient to adopt the following definitions :

Two directions are said to be the same when they are **parallel** *and* **in the same sense** *(though* **not** *necessarily in the same straight line) ; two directions which are* **parallel** *but* **not in the same sense** *are said to be* **opposite.**

Thus, if two ships start from different points on the equator

and both move Northwards or both Southwards, they are moving in the same direction.

Two bodies are said to have the same or equal velocities when their velocities are equal in magnitude and in the same direction.

In accordance with this definition, a velocity will be completely determined when we know its magnitude and direction; a full knowledge of the velocity gives us the magnitude and direction of the motion, but tells us nothing as to the position of the body or the actual path along which it is travelling.

Finding it necessary to distinguish something more than the mere speed of moving bodies, we have extended our definition of velocity so as to include the direction of motion, and have thus provided ourselves with a convenient name for a quantity which is constantly occurring in dynamical problems. At the same time we have been careful to say exactly what we now mean by *velocity*; how much the term includes, and how much it does not include.

Displacement is to be defined in a perfectly similar manner. *A displacement will be determined by a given* **length** *measured in a given* **direction**; *but the term has* **not** *any reference to the* **actual line** *along which this length is measured.*

51. **Composition of Two Displacements.**—Let a particle be placed at the corner of the parallelogram O X P Y (fig. 15), and

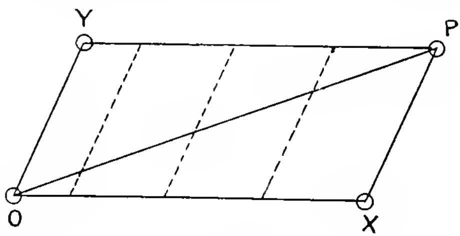


FIG. 15.

let it receive the displacement whose magnitude and direction are specified by O X or Y P; it will then be at X. Replace the particle at O, and let it receive the displacement O Y or X P; it will now be at Y. Once more replace the particle at O, and let it receive successively the two displacements O X and

O Y. First, the displacement O X will bring the particle to X; then the displacement O Y (or X P) will bring it from X to P. Had these same displacements been given to the particle in the reverse order (*i.e.* first O Y and then O X) the particle would have moved from O to Y and from Y to P; but in either case the whole effect, or *resultant* effect, of the two displacements has been to transfer the particle from O to P; O P being therefore called the *resultant displacement.*

Hence the proposition: *If two displacements be represented in direction and magnitude by two straight lines drawn* **from** *a point, and if a parallelogram be constructed having these two straight lines for adjacent sides; then the resultant displacement will be represented in direction and magnitude by the diagonal of the parallelogram which is drawn* **from** *that point.*

Observe, we took the component displacements to be O X, O Y (drawn *from* O), *not* X O, Y O; and, accordingly, the resultant displacement is O P (drawn *from* O) *not* P O.

52. To render our ideas more definite, let a straight wire coincide with O X, and let the movable particle be a bead threaded on the wire and capable of sliding along it. If the wire moves into the position Y P, every point in it has received a displacement equal to O Y. If the bead moves from end to end of the wire in the direction O X, the bead will have an *additional* displacement O X; and no matter whether the wire itself moves first or the bead moves first upon the wire, or whether the two movements take place simultaneously, the resultant displacement will be the same; for when the wire has reached the position Y P, and the bead has reached the further end of the wire, the position of the bead will be P. This proposition may be extended to any number of displacements. Suppose there is a pile of n books on a flat table; then n separate displacements may be given to the top book as follows: First the top book alone receives a displacement D_1; (of course D_1 includes the direction as well as the magnitude of the displacement), and this determines the relative positions of the first and second books. Now let the first and second books together receive a displacement D_2; this determines the relative positions of the second and third books. Next, we move the

three top books through a displacement D_3, and so on ; until, finally, the whole pile of books is pushed along the table by an amount D_n, which determines the position of the last (or n^{th}) book relative to the table. Now, the top book has received successively these n displacements, D_1, D_2 ... D_n, and each of these displacements has determined the relative positions of two consecutive books (or between the lowest book and the table). Retaining the *same meanings* for D_1, D_2, &c. (so that, for instance, D_2 always means the displacement given to the first two books alone), it is obvious that the position on the table of the last book, and the *form* of the pile of books, will be fixed completely by the separate values of D_1, D_2 ... D_n, and will be independent of the order in which these displacements occur. The final position of the top book, then, will be independent of the order in which it receives the component displacements D_1, D_2 ... D_n.

53. **Resultant of any number of Independent Displacements in One Plane.**

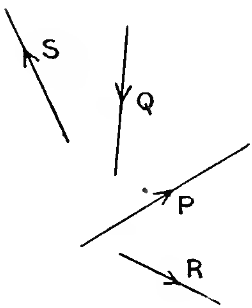


FIG. 16.

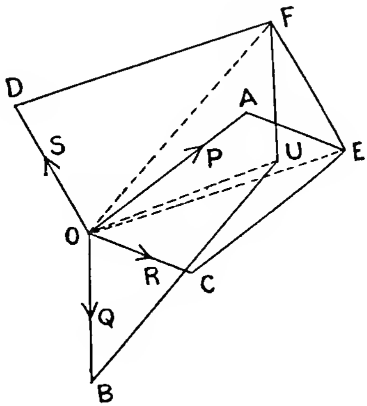


FIG. 17.

Let the component displacements be P, Q, R, S (fig. 16), where each of the lines is marked with an arrow-head, so as to leave no ambiguity as to the *direction* of the displacement. In accordance with the first definition in § 50, any straight lines

equal to P, Q, R, S, and drawn in the same directions, will represent the same displacements; and by the proposition of § 52 we may compound the displacements P, Q, R, S, in any order we please.

Take any point O (fig. 17) in the plane of P, Q, R, S, and draw O A = P, and in the same direction; and, similarly, draw O B, O C, OD *equal* respectively to the *displacements* Q, R, S. Compounding O A and O C by the parallelogram of displacements, the resultant is O E. Compounding O E and O D, the resultant is O F; compounding O F and O B, the resultant is O U. This displacement O U, which is the resultant of all the component displacements, P, Q, R, S, will have the same magnitude and direction in whatever order the composition is effected.

54. **Polygon of Displacements.**—A somewhat different method of compounding displacements will often be found more convenient. Commencing with a particle at a given point O (fig. 18), we give to this particle each of the displacements in succession, the *order* in which the displacements occur being a matter of indifference. Take, for example, the order P, Q, R, S. Draw O A equal to the displacement P; the displacement P will therefore bring the particle from O to A. From A draw A B = Q; then the displacement Q will bring the particle from A to B. From B draw B C = R, and from C draw C D = S; then the successive displacements R and S will move the particle from B to C and from C to D.

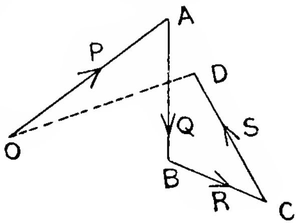


FIG. 18.

What, then, has been the resultant effect of the successive displacements P, Q, R, S? The particle has moved from O to D; hence, O D (*not* D O) is the resultant displacement. On comparing figs. 16 and 17 it will be seen that the resultant O U found by the first method is identical in direction and magnitude with the resultant OD found by the second method.

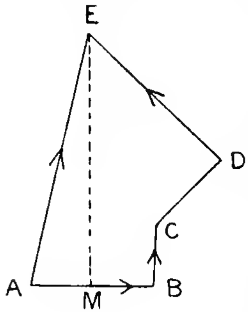
55. It is obvious that the successive journeys of the particle must form a continuous path, such as O A B C D; and there-

fore, since P, Q, R, S are successive displacements, they must be represented by O A, A B, B C, C D; *not* (for example) by O A, B A, B C, C D. For if the first displacement has brought the particle to A, A *must be the starting-point for the next journey*, and consequently the next portion of the path will be the line A B, drawn **from** A.

Hence the proposition: *To find the resultant of any number of displacements, let the displacements be considered in any order whatever; and having selected some fixed point* O, *draw* **from** *this fixed point a straight line equal to the first displacement;* **from** *the end of this, another straight line equal to the second displacement;* **from** *the end of this, another equal to the third displacement, and so on, until one such line has been drawn equal to each displacement. The straight line drawn* **from** *the point* O **to** *the extremity of the last line, is equal to the resultant of all the displacements.*

Examples.

(1) A man has been walking for $2\frac{1}{2}$ hours at a uniform speed of 3 miles per hour. During 40 minutes of the time he has been walking in a due easterly direction; during 20 minutes in a due northerly direction; during 30 minutes his course has been north-east; and during one hour, north-west. Find the direction and magnitude of his distance from the starting-point.



In 40 minutes he walks 2 miles; in 20 minutes 1 mile; in 30 minutes $1\frac{1}{2}$ miles; and in 1 hour 3 miles. Thus the displacements to be compounded are: 2 miles east, 1 mile north, $1\frac{1}{2}$ miles north-east, and 3 miles north-west.

Draw A B, B C, C D, D E, proportional in magnitude to these respective displacements, and in the same directions. Then A E will represent in direction and magnitude the resultant displacement.

Since C D $(=)\ \frac{3}{2}$ miles and D E $(=)$ 3 miles, the easterly component of C D $(=)\ \frac{3}{2}.\cos 45^\circ\ (=)\ \frac{3}{2}\cdot\frac{1}{\sqrt{2}}$ miles, and the easterly

component of D E (=) − 3, cos 45° (=) − 3 . $\frac{1}{\sqrt{2}}$ miles. Thus A M, the easterly component of A E (=) 2 + $(\frac{3}{2} - 3)\frac{1}{\sqrt{2}}$ miles (=) $\frac{8 - 3\sqrt{2}}{4}$ miles.

Similarly, M E, the northerly component of A E, is found to be (=) 1 + $(\frac{3}{2} + 3)\frac{1}{\sqrt{2}}$ miles (=) $\frac{4 + 9\sqrt{2}}{4}$ miles.

Thus the length of A E, being equal to $\sqrt{AM^2 + ME^2}$, is known; and to determine the direction of A E, we have tan B A E = M E / A M.

(2) If a number of displacements can be represented in direction and magnitude by the sides of a closed polygon *taken in order*, the resultant of all the displacements is zero; and, conversely, if the resultant is zero, the displacements may be represented by the sides of a closed polygon taken in order.

NOTE.—A *closed polygon* is a figure bounded by a number of straight lines, of which the last terminates where the first begins; and the words '*taken in order*' indicate that the sides are to be considered in the same directions as in moving *continuously* around the perimeter. Thus in fig. 18, O A B C D O is a closed polygon, and its sides, *taken in order*, would be O A, D O, B C, C D, A B, whose directions are consistent with the continuous path O A, A B, B C, C D, D O; or else A O, O D, C B, D C, B A, corresponding to a continuous circuit in the opposite direction. The essential point is *not the order* in which the actual displacements are considered; but *the directions of the several displacements*, and these must be consistent with a continuous circuit of the perimeter.

The student will find no difficulty in proving the proposition, which is sometimes called the **Polygon of Displacements.**

56. **Resolution of Displacements.**—*Any given displacement may be resolved into two components having given directions.*

For example (fig. 19), to find the components in the directions K L, M N of the displacement O P: draw P A parallel to K L and O A parallel to M N, meeting P A at A; also draw O B parallel to K L and P B parallel to M N, meeting O B at B.

Then O A P B is a parallelogram, and by the parallelogram of displacements, O P is the resultant of the displacements O B and O A; that is, O B and O A are the components of O P, *parallel* to the given directions K L, M N. It will be seen that in

the present case O B is in the *same* direction as K L, while O A is in the *opposite* direction to M N ; so that the resolution of O P in the given directions yields a *positive* component in the direction K L and a *negative* component in the direction M N.

It must be observed that in general the directions of *both* components must be fixed before we can determine the magnitude of either. To realise this, let the directions for resolving O P be K L′, M N, and draw P A′, O B′, parallel to K L′, as in the figure. Then O P will be resolved into the displacements O A′ and O B′ ; and though M N is still one of the directions of resolution, the component in this direction is no longer the same.

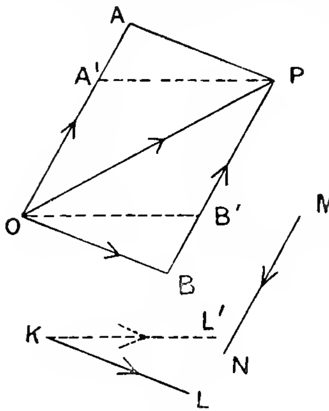


FIG. 19.

57. **Rectangular Components.**—There is one case which is of special importance on account of its simplicity and the frequency of its occurrence in practice, namely, when a displacement has to be resolved in two directions at right angles to one another. Let O P be the given displacement, and let O M, P M be drawn parallel to the directions in which O P is to be resolved.

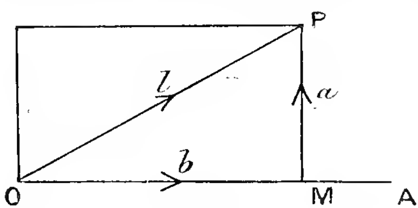


FIG. 20.

Then O M P is a right angle, and O M, M P are rectangular components of the displacement O P. From § 56 we know that the component in the direction O M depends on the direction chosen for the other component ; but when the component in the direction O M is spoken of without further

qualification, it is understood that the second component is perpendicular to O M. From this it immediately follows that the component of any displacement in a direction at right angles to itself is zero, while the component in the direction of the displacement is equal to the displacement itself. If we require to find the component of a displacement O P resolved in a direction O A (fig. 20 or fig. 21), we let fall the perpendicular PM on O A (or AO produced), and O M is the component sought for. O M is reckoned positive if it lies in the same direction as O A, negative if in the opposite direction. In either case O M / O P is the cosine of the angle A O P, and A O P is the angle between the direction of the displacement and the direction of resolution; that is

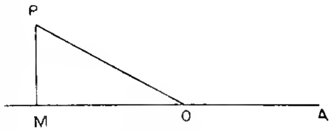


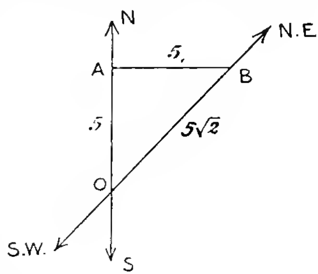
Fig. 21.

$$O\,M = O\,P \cos A\,O\,P \qquad\text{.....................}(\mathbf{10})$$

or, *The component of a displacement resolved in any direction is equal in magnitude to the displacement multiplied by the cosine of the angle between the direction of the displacement and the direction of resolution.*

Example.

There are two straight intersecting roads, one running north and south, the other north-east and south-west; and a station on the first is 5 miles due west of a station on the second. Find the positions of the two stations.



Let A and B be the positions required; then the problem amounts to finding the components of A B resolved in a northerly and in a north-easterly direction.

Since A is on the first road, the position of this latter may be fixed by drawing a straight line A O north and south through A; similarly, the position of the second road will be determined by a straight line north-east and south-west through B; and the intersection O of A O and B O will determine the intersection of the roads.

From the figure it is evident that A is 5 miles to the north and B $5\sqrt{2}$ miles to the north-east of O, and this determines the positions of the two stations. If we wished to walk from B to A by way of the two roads we should have to walk $5\sqrt{2}$ miles to the south-west, and then 5 miles to the north; the resultant of these two displacements being 5 miles to the west.

Examples on Chapter V.

(1) A ladder 10 metres long rests at an inclination of 60° to the horizontal; if a man *ascends* the whole length of the ladder, find the resolved part of his displacement in a *vertically downward* direction.

(2) A body is conveyed from the North Pole to a point on the equator; what is the component of its displacement in a direction parallel to the earth's axis?

(3) A wheel rolls along the ground through a distance equal to half its circumference. Show how to find the resultant displacement of any point on the rim of the wheel, and hence determine which point has the greatest displacement and which the least.

The wheel in rolling this distance turns through two right angles, and the displacement of each point is made up of two components, one due to the translation of the wheel, the other due to its rotation.

(4) The minute hand of a clock is 15 centimetres long, and the hour hand 10 centimetres; supposing the clock to indicate the correct time, find in direction and magnitude the displacement of the end of the minute hand relatively to the end of the hour hand, between the hours of three and half-past six.

(5) A ship has moved straight ahead through 7 metres in a north-easterly direction. How must a passenger change his position on deck that his total displacement may be eight metres to the north?

(6) A man walks three miles in a straight line, and then, changing his course by 60°, walks another six miles. Find in direction and magnitude his total displacement.

(7) At a certain instant a fly is at the hub of a cart-wheel, and one of the spokes is pointing vertically upward. The fly

crawls to the end of this spoke while the wheel turns round $2\frac{3}{4}$ times owing to the motion of the cart. What is then the total displacement of the fly?

(8) A has walked 5 miles to the north, and B, having walked $5\sqrt{2}/2$ miles to the north-west, is now 10 miles to the south-west of A. What was originally the distance from B to A?

(9) Resolve a westerly displacement of 12 metres in two directions which are respectively southerly and south-easterly.

(10) Resolve a displacement in two directions, each inclined 60° to the direction of the displacement.

CHAPTER VI

COMPOSITION OF VELOCITIES AND ACCELERATIONS

58. **Composition of Two Velocities.**—Having now learned something about the composition of independent displacements, let us inquire what is meant by a body having simultaneously two independent velocities. At any instant, a body if moving must be moving in some one definite direction, and its velocity must have some definite magnitude; though we may regard this velocity as made up of two or more independent components. In § 19 the passenger walking along the floor of the tram-car has but one velocity relative to the road, though this velocity is the sum of two independent components; the velocity of the passenger himself relative to the car, and the velocity of the car relative to the road.

When the component velocities are not along the same straight line, we may use the same mechanical illustration of a bead moving along a wire which has previously served (§ 52) for the composition of displacements; but a few words first as to the

59. **Representation of Velocities.**—A uniform velocity is completely specified by the *displacement per unit time*; a displacement being a quantity which has direction as well as magnitude. We may therefore conveniently *represent* the velocity of a uniformly moving body by drawing from any point a straight line equal to the displacement received by the body in one unit of time. Thus, a velocity of one metre per second vertically upwards may be *represented* by a straight line one metre long drawn vertically upwards from any point; or if the diagram thus constructed is too large or too small for practical purposes, a given velocity may be equally well represented by

the displacement produced in a shorter or a longer time—say in $\frac{1}{200}$ of a second, or in one hour—but of course every velocity throughout the same diagram must be represented according to the same scale.

60. Let there be a straight wire occupying the position O X, and moving parallel to itself [1] with uniform velocity in the direction O Y (fig. 22). Suppose that after an interval of time, t, the wire has moved into the position Y P; we may take O Y or X P to represent the velocity of the wire, and the scale of our diagram is thus fixed by the value of t. Further, while the wire is in motion, suppose that there is a bead sliding uniformly along the wire from left to right with such a velocity that in the time t its displacement along the wire is equal to O X; then O X or Y P will represent, *on the scale of our diagram*, the velocity of the bead relative to the wire.

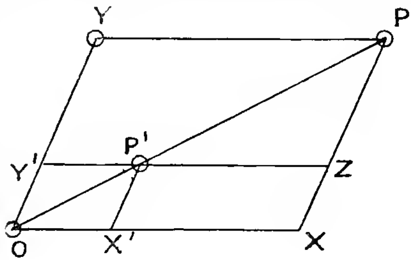


FIG. 22.

Starting, then, with the wire in the position O X and the bead at O, it will be found after the time t that the wire is in the position Y P and the bead at the further end of the wire, that is, at P. Now, on our diagram, a uniform velocity which produces a displacement O P in the time t is *represented by* O P; moreover, the wire moves with uniform velocity parallel to itself, and the bead moves with uniform velocity along the wire; but does the combination of these motions produce a uniform velocity in the bead? Yes; for starting once more with the wire in the position O X and the bead at O, consider the displacements which occur in the time $\frac{t}{n}$. Let the displacement of the wire in this time $=$ O Y′, and that of the bead

[1] That is, the motion of the wire is one of pure translation; every subsequent position, such as Y′ Z or Y P, being parallel to O X.

along the wire $= Y' P'$; so that the wire is now in the position $Y'Z$, parallel to OX, and the bead is at P'. Since the component velocities along OX and OY are *uniform*, $OY' = \frac{1}{n} OY$ and $OX' = \frac{1}{n} OX$; the displacements being proportional to the times in which they are produced. It remains to be proved that P' lies on the straight line OP, and that $OP' = \frac{1}{n} OP$. Draw $P'X'$ parallel to YO, meeting OX in X'. Then $X'P' = OY' = \frac{1}{n} OY$; and in the two triangles $OX'P'$, OXP we have $OX' = \frac{1}{n} \cdot OX$ and $X'P' = \frac{1}{n} \cdot XP$, and the angle $OX'P' =$ the angle OXP; hence, these two triangles are similar, and $OP' = \frac{1}{n} OP$, and the angle $X'OP' =$ the angle XOP; so that P' lies on OP.

Hence, the path of the bead is along the straight line OP, and the space described in any interval of time is proportional to that interval. The velocity of the bead, then, is constant in direction and magnitude, and is therefore represented (on the scale of our diagram) by OP. This establishes the proposition known as the

Parallelogram of Velocities.—*If two given velocities be represented in direction and magnitude by two straight lines drawn* **from** *a point; and if a parallelogram be constructed having these two straight lines for adjacent sides; then the resultant of the two velocities existing simultaneously will be represented in direction and magnitude by the diagonal of the parallelogram which is drawn* **from** *that point.*

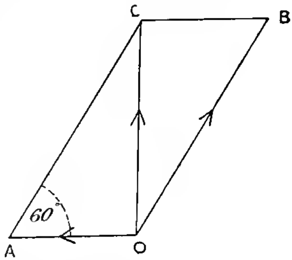
Examples.

(1) The velocity of a body is made up of two components, whose directions are inclined at 120° to one another, and one of the components has twice the magnitude of the other. Find the resultant velocity.

Let OA represent one of the components (v), and OB the other component ($2v$), the angle BOA being equal to 120°. Then the

diagonal O C of the parallelogram A O B C represents the resultant velocity.

Now, the angle O A C, being the supplement of B O A, is equal to $60°$, and since $A O = \frac{1}{2} A C$, the triangle C A O is evidently half of an equilateral triangle, C O A being a right angle and O C A an angle of $30°$.



Thus $O C^2 = A C^2 - A O^2 (=) (2v)^2 - v^2 = 3v^2$;

$$\text{or, } O C (=) \sqrt{3}\, v.$$

The resultant velocity is thus equal in magnitude to $\sqrt{3}$ times the smaller velocity and is perpendicular to it in direction.

(2) Two trains are running with uniform velocities on lines at right angles to one another; one at 20, the other at 48 miles per hour. Find the *apparent* motion of the second train as seen from the first.

If the second train were at rest it would *appear* to have a velocity equal and opposite to the actual velocity of the first. If the first train were at rest the true motion of the second train would be apparent to the observer.

When both trains are moving, the apparent motion of the second must be found by compounding its actual velocity with a velocity equal and opposite to that of the first.

61. **Superposition of Motions.**—In order to simplify the investigation, the velocities have been taken as uniform, but it is easy to extend the proposition to the case of variable velocities. Instead of a straight wire, take a wire O A bent into any form, and let there be a bead which can slide along this wire. By giving a suitable form to the wire, we can make the bead follow any path we please, and we may suppose the rate of motion of the bead to vary in any manner as it travels along the wire. Further, let the wire have an independent motion of its own, so that its extremity describes the path O Y B with variable speed; and let the motion of the wire be one of pure translation (§ 7), so that all points of the wire describe exactly equal and similar paths (indicated by the

dotted lines), and at any given instant they have all the same velocity, *namely, the velocity of the wire at that instant.*

Starting at the instant when the wire is in the position O A and the bead at the extremity O of the wire, consider what will have happened in the time *t*. The wire will have moved into some new position Y A′, and at the same time the bead will have moved along the wire, so that it occupies some position P on Y A′. Now, it is evident that the bead will have arrived at the position P, whether we first move the bead along the wire with proper regard to speed during the time *t*, and then let the wire move along the path O B during an equal interval of time, or whether these two movements occur in the opposite

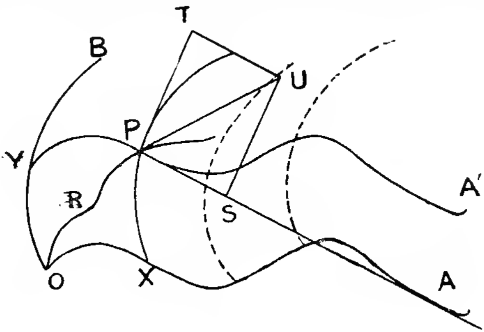


FIG. 23.

order, or whether they take place simultaneously. In the first case, the *path* of the bead will be O Y P, in the second case O X P ; and when the two motions occur simultaneously, the bead will describe some intermediate path, such as O R P, the velocity *at each instant* being the resultant of two components : the velocity of the bead *along the wire* and the velocity of the wire itself.

We have, therefore, established this important proposition :

If during a time t *a body is subject to two independent motions, whether uniform or variable, the actual displacement of the body will be the same as if each of the motions had taken place separately, each during an interval of time equal to* t.

This result may easily be extended to any number of inde-

pendent motions; for we may suppose the path O B to be a wire along which the extremity of the wire O A moves. We may then give any motion of *pure translation* to the wire O B, and this will constitute a third independent motion; the resultant displacement of the bead during any time t, being still independent of the simultaneity or order of succession of the independent motions. So we may proceed to any number of independent motions.

62. Returning to the more simple case of the wire and the bead in fig. 23, consider the instant of time when the wire is in the position Y A′, so that the bead, by travelling along the path O R P, has arrived at P. The velocity of the bead at this instant is the resultant of two component velocities; one of which is due to the motion of the bead along the wire, while the other is due to the motion of the wire itself. At the point P draw P S, a tangent to the wire Y P A′, and P T, a tangent to the path X P, along which the point P of the wire is moving. The velocity of the wire, then, is in the direction P T, and the velocity of the bead relatively to the wire is in the direction P S. If the portion of the wire beyond P were straightened out in the direction P S, and if the velocity of the bead along the wire became uniform at the instant we consider, the displacement P S along the wire, accomplished in the time τ, would represent (according to a certain scale) the velocity of the bead along the wire. Similarly, if from the instant considered the velocity of the wire became uniform, so that the subsequent motion was along P T, the point P of the wire would, after the time τ, have arrived at some point T, so that P T represents (on the same scale as before) the velocity of the wire at the instant.

Complete the parallelogram P T U S and join P U. Then P U represents the velocity which the bead would have if both the components of its motion became uniform at the instant considered; that is, P U represents the value of the *resultant* velocity *at the instant* when the *component* velocities are represented by P S and P T. The parallelogram of velocities is, therefore, seen to apply *at each instant* when the component velocities are variable.

63. Any number of velocities may be compounded by successive applications of the parallelogram law. Any two of the velocities may first be compounded, the resultant of these two being then compounded with one of the remaining velocities, and so on, or we may apply the proposition known as the

64. **Polygon of Velocities.**—It has already been shown in treating of displacements that the polygon construction is *geometrically equivalent* to a series of parallelogram constructions; but we may also give to it a rather more direct interpretation. Whether the component velocities be uniform or

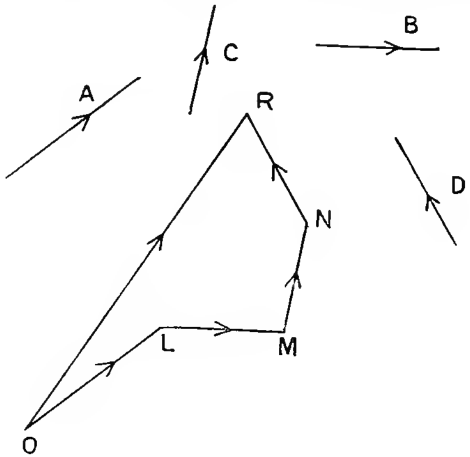


FIG. 24.

whether they vary in direction and magnitude, let us choose some particular instant, and let the values of the component velocities *at that instant* be represented in direction and magnitude by A, B, C, D (drawn in the sense indicated by the arrows). Draw O L, L M, M N, N R, respectively equal to, and in the same direction as, A, B, C, D. Then O L is equal to the displacement which would be produced by the velocity (=) A alone, if it continued uniformly during a certain time τ (the value of τ being determined by the scale of the diagram). L M is the displacement which would be produced by the

velocity (=) B alone if continued uniformly during an equal period τ, and so on. Hence, if each of the velocities A, B, C, D, is given in turn to a body, and if each velocity is continued uniformly for a period equal to τ, the total displacement of the body will be O R. Therefore (from the proposition of § 61), if the velocities A, B, C, D, were given simultaneously to the body, and continued to act uniformly together for the period τ, they would produce a displacement equal in magnitude and direction to O R. O R, therefore, represents the resultant velocity on the scale of our diagram. The *Polygon of Velocities* may be enunciated as follows :

To find the resultant of any number of velocities, let the velocities be considered in any order whatever, and, having selected a fixed point O, *draw* from *this fixed point a straight line representing the first velocity*, from *the extremity of this, another straight line representing the second velocity ; and so on, until one such line has been drawn to represent each velocity. The straight line drawn* from *the point* O to *the extremity of the last line, represents the resultant of all the velocities.*

65. An important case occurs when the system of lines O L, L M, etc. (fig. 24), forms a *closed polygon* ; that is, when the last line of the system terminates at the point where the first line began. Take, for example, the velocities represented by O A, A B, B C, C D, D E, E O (fig. 25). The resultant of these is represented by the straight line drawn from O to O; that is, the resultant velocity is zero, the body being, therefore, without motion at the instant considered. Another way of looking at the question is this : The resultant of the velocities represented by O A, A B, B C, C D, D E, is represented by O E ; hence, if we add to the system a velocity (=) E O we shall reduce the resultant

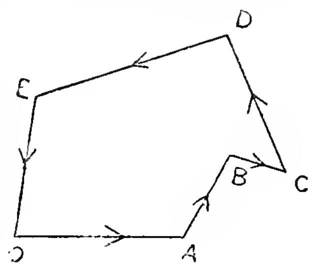
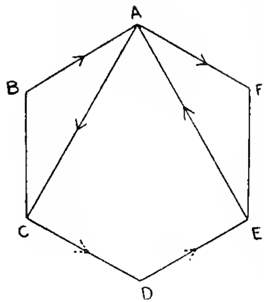


FIG. 25.

velocity to zero (for the velocities represented by O E and E O are equal and opposite). Or again, if the velocities represented by O A, A B, B C, C D, D E, E O, were given to the body *successively*, each continuing uniformly for a time equal to τ, the successive displacements of the body would be O A, A B, B C, C D, D E, E O ; the resultant displacement being zero. Hence, if all the velocities had acted simultaneously, and had continued uniform for the time τ, they would have produced no displacement. Their resultant is accordingly zero.

Examples.

(1) A B C D E F is a regular hexagon, and the velocity of a certain body is made up of components proportional in magnitude to B A, A F, E A and A C respectively, and in the same directions. Show that on the whole the body is without motion.



Since B A and A F are respectively equal to D E and C D and in the same directions, the component velocities may be represented in direction and magnitude by D E, C D, E A, A C ; and these straight lines, taken each in its proper direction, form a closed path A C, C D, D E, E A. The resultant velocity is therefore zero.

(2) The velocity A relative to B is 10 centimetres per second northward ; the velocity of B relative to C, 15 centimetres per second westward ; of C relative to D, 12 centimetres per second southward ; and of D relative to E, 13 centimetres per second eastward. Find, by means of a diagram the velocity of A relative to E.

66. **Resolution of Velocities.**—Since the composition of velocities follows exactly the same laws as the composition of displacements, the laws of resolution must also be the same for each. Thus, if O P represents a given velocity in direction (fig. 26), we may draw O X and P X (or P Y and O Y) respectively parallel to two given directions, and the velocity represented by O P may thus be resolved into two components ; one (=) O X or Y P, the other (=) O Y or X P ; each such com-

ponent being affected with the positive sign if it lies in the same sense as the corresponding direction of resolution, and with the negative sign if it lies in the contrary sense.

If the directions of resolution are perpendicular to one another, it may be shown exactly, as in § 57, that the component of a velocity v, resolved in a direction inclined at angle θ to this velocity is equal to

$v \cos \theta$......................... (11)

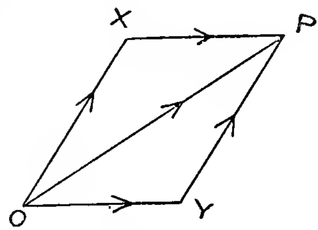


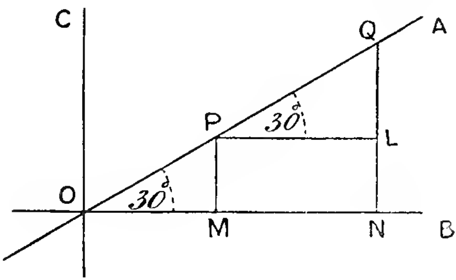
FIG. 26.

Examples.

(1) Two lines of railway are inclined to one another at an angle of 30°, and a train on one line is moving away from the junction at the rate of 10 metres per second. At what rate does its perpendicular distance from the second line increase?

Let O A be the line on which the train is moving, and O B the other line; the angle B O A being equal to 30° (see fig. below).

The rate at which the train recedes from O B is evidently the resolved part of its velocity perpendicular to O B; for, let P be the



position of the train at a given instant, and Q its position one second afterwards, so that P Q = 10 metres. Let fall the perpendiculars P M, Q N on O B, and the perpendicular P L on Q N.

Then, during one second, the train's perpendicular distance from O B has increased by L Q; and since L P Q = 30°, and P Q L, its complement, = 60°, L Q / P Q = cos 60° = $\frac{1}{2}$, and the velocity sought for is $\frac{10}{2}$, or 5 metres per second.

This may help to make clear what is meant by the resolved part of the train's velocity perpendicular to O B; and, similarly, the resolved part of the velocity in the direction O B is the rate at which the train recedes from O C; that is, P L / P Q × 10 metres per second, or $5\sqrt{3}$ metres per second.

(2) Two given directions are inclined to one another at an angle of 120°, and the direction of a certain velocity (v) makes with them angles of 60° and − 60°. Find the components of v resolved in the given directions.

67. **Composition of Accelerations.**—We have seen that displacement, velocity, and acceleration are all directed quantities; that in order to specify completely a displacement, a velocity, or an acceleration, we must define its direction as well as its magnitude; and it has further been pointed out (§ 13) that acceleration is related to velocity in the same way that velocity is related to displacement: velocity being rate of change of displacement, and acceleration being rare of change of velocity. Having found, then, that precisely the same laws of composition and resolution apply to displacements and to velocities, we might be led to expect these laws were also applicable in the case of accelerations, and it will now be shown that such is actually the case.

Let F, G, H . . be any number of uniform accelerations impressed on a body, all of which are along the same straight line, though *not* necessarily all in the *same direction*, so that some of the quantities F, G, H . . . may be positive while others are negative. If F were the only acceleration, the velocity of the body would increase *in each unit of time* by F units of velocity; if G were the only acceleration, the increase of velocity *in the unit of time* would be G, and so on. When all the accelerations F, G, H . are present simultaneously, the *increase of velocity in each unit of time* will be $F+G+H+$ . . that is, $F+G+H+$ . . is the measure of the resultant acceleration.

68. If the component accelerations are not uniform, let F, G, H . . be their respective values *at the instant considered* and let τ be an *extremely short* interval of time immediately following that instant. Then the acceleration F, if acting by

itself for the time τ, would increase the velocity of the body by $F\tau$ units of velocity . . . and so on; and if the accelerations F, G, H . . . all act simultaneously, they will increase the velocity during the time τ by $(F+G+H+ \ldots)\tau$ units of velocity; that is, $F+G+H+\ldots$ will measure the resultant acceleration at the instant considered.

If $F+G+H+\ldots$ is a positive quantity, the resultant acceleration is in the direction we have chosen to call positive; if $F+G+H+\ldots$ is negative, the resultant acceleration is in the contrary direction. This, of course, tells us nothing as to the present value of the *velocity*, which may have any positive or negative value quite independently (§§ 15, 16).

69. To realise more clearly what is meant by two accelerations acting simultaneously, let us return once more to the case of a bead sliding on a straight wire, while the wire at the same time moves parallel to itself; and, instead of supposing the sliding-velocity of the bead to be constant, let it be subject to a uniform acceleration along the wire, and at the same time let the motion of the wire parallel to itself be uniformly accelerated.

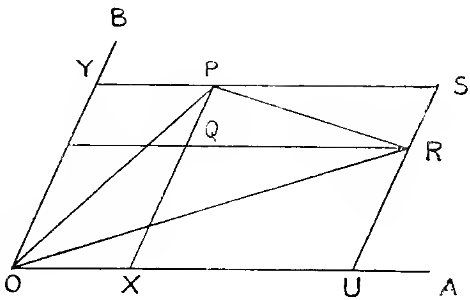


FIG 27.

Let O A (fig. 27) be a straight line to which the wire is always parallel, and let every particle in the substance of the wire describe a rectilinear path parallel to O B. Let O X represent (according to some convenient scale) the sliding-velocity of the bead at a given instant, and let O Y represent (according to the same scale) the simultaneous velocity of the wire.

Then the resultant velocity of the bead *at the given instant* is represented by the diagonal O P of the parallelogram O X P Y. As time goes on, however, the two components of the bead's velocity will be subject to continuous change, and at the end of an interval, t, we shall find a new value for the resultant.

Let the positive direction along O A be from O to A, and the positive direction along O B from O to B. Let F (measured in the direction O A) be the acceleration of the sliding motion, and G (measured in the direction O B) the acceleration of the motion of the wire.

At the end of the time t the sliding-velocity will have increased by F t units; the velocity of the wire by G t units; and, adhering still to the scale of our diagram, we may draw P S to represent the velocity F t, and P Q to represent the velocity Gt. On completing the construction of fig. 27, it will be seen that P Q is drawn in the negative direction, while the initial velocity (=) O Y is in the positive direction, that is, in the case we have chosen, the speed of the *wire* happens to be decreasing.

Now, at a certain instant the resultant velocity of the bead was (=) O P, and during the time t two new components of velocity have been added, reprcsented respectively by P S and P Q; the whole velocity added in the time t will, therefore, be represented by P R, and the whole velocity of the bead at the end of the time t will be represented by O R, which is the resultant of O P and P R.

But P S represents in direction and magnitude the velocity added in the time t by the acceleration F; P Q represents the velocity added in the time t by the acceleration G; and P R the velocity added in the same time by the joint action of the two accelerations. Hence, P S may be taken to *represent* the acceleration F according to a new scale and P Q will then represent the acceleration G according to the same scale, while P R will represent the resultant acceleration. This proves the proposition known as the

Parallelogram of Accelerations.—*If two accelerations be represented in direction and magnitude by two straight lines drawn* **from** *a point, and if a parallelogram be constructed having these two straight lines for adjacent sides, the resultant*

of the two accelerations will be represented in direction and magnitude by the diagonal of the parallelogram which is drawn **from** *that point.*

70. The student will have noticed that P S and P Q (fig. 27) were at first spoken of as representing velocities and later as representing accelerations, *but according to a different scale.* Now, the *scale* for measuring *velocities* is an *interval of time*, which we may call T, the velocity v being represented by a length v T drawn in its own direction; and a straight line on the diagram represents that velocity with which a point must move to describe that straight line uniformly in the time T. The *scale* for measuring *accelerations* must be the *product of two intervals of time*, say T and t; a straight line being made to represent an acceleration in the following way: the *acceleration* acting uniformly for the time t would produce a certain *velocity*, and *this velocity*, continuing uniformly for the time T, would produce a *displacement* of the same direction and magnitude as the straight line in question.

71. The parallelogram of displacements was established by geometry (§§ 51, 52);

From this followed the parallelogram of velocities, by considering the component and resultant displacements described in an interval of time (§ 60);

And from this, again, the parallelogram of accelerations, by considering the component and resultant velocities produced in a given interval.

72. If a point be simultaneously subject to three or more independent accelerations, these may be compounded by successive applications of the parallelogram law, or we may use the polygon construction, which is known to lead to exactly the same results.

The **Polygon of Accelerations** is enunciated as follows: *To find the resultant of any number of accelerations, let the accelerations be considered in any order; and having selected a fixed point* O, *draw* **from** *this point a straight line representing the first acceleration;* **from** *the extremity of this a straight line representing the second acceleration; and so on, until one such line has been drawn to represent each acceleration. The*

straight line drawn **from** *the point* O **to** *the extremity of the last of these lines represents the resultant of all the accelerations.*

73. It will be evident that if a number of accelerations act simultaneously on a moving point during a given time, *the same change of velocity* will be produced as if the accelerations had acted separately, each for an equal time; for the resultant change of velocity is obtained by compounding the changes due to the several accelerations, and is independent of the order of composition.

74. In § 61 it was shown that when a body is simultaneously actuated by several independent velocities, the *displacement* described in any time is the same in direction and magnitude as if each of the velocities had acted separately for an equal time. It must be observed, however, that the *path* followed by the body will depend on the order in which the motions take place, the *destination* only being the same in each case.

And again, in § 73 above, both the *path* and the *destination* will depend on the order of the accelerations, though the *final velocity* is in each case the same.

75. Although the same laws of composition hold good for displacements, for velocities, and for accelerations, these three kinds of quantities are, as already stated, quite distinct. Displacements cannot be compounded with velocities or velocities with accelerations; but an acceleration *continuing for an interval of time* produces a change of velocity, and this must be compounded with the initial velocity to obtain the final velocity; in the same way a velocity *continuing for an interval of time* produces a certain displacement, and this must be compounded with the initial displacement to obtain the final displacement.

Examples.

(1) A body is moving freely under the action of gravity; find the resolved part of its acceleration in a direction inclined 30° to the horizon.

(2) If the acceleration of a body is $g\sqrt{2}$ in a downward direction inclined 45° to the vertical, what acceleration is there besides that due to gravity?

EXAMPLES ON CHAPTER VI.

(1) Show that if three velocities (or accelerations) are represented in direction and magnitude by the sides of a triangle, taken in order, their resultant will be zero.

This proposition is called the **triangle of velocities** (or of **accelerations**).

(2) Prove the converse of the preceding proposition; and hence show that if the resultant of three velocities (or accelerations) is zero, and that if a triangle be drawn whose sides taken in order have the same directions as these three velocities (or accelerations), each side will be proportional in magnitude to the corresponding velocity (or acceleration).

(3) What acceleration besides that due to gravity must be impressed on a body that its resultant acceleration may be horizontal and equal in magnitude to g?

(4) A body begins moving in a given direction with a velocity of 500 centimetres per second, and is subject to an acceleration of $100\sqrt{2}$ centimetres per second per second in a direction inclined 45° to the first. What will be the velocity after 2·5 seconds?

(5) Find the resultant of two velocities, one of which has twice the magnitude of the other, and is inclined to it at an angle of 60°.

(6) An ocean current is flowing northwards at the rate of 2 metres per second; a ship's velocity *relative to the water* is 5 metres per second eastward, and a sailor is climbing up a mast of the ship at the rate of 1 metre per second. Find the velocity of the sailor relative to the sea-bottom.

(7) A body is subject to an acceleration which is constant in direction and magnitude; its velocity at a certain instant is 10 centimetres per second; and, after 5 seconds, the velocity has the same magnitude as at first, but is perpendicular to its former direction. What is the value of the acceleration?

(8) Resolve an acceleration of 50 centimetres per second per second into two mutually perpendicular components, of which one is inclined at 30° to the direction of the given acceleration.

(9) Find the resultant of two accelerations whose magnitudes are respectively f and $2f$, and whose directions make an angle of 120° with one another.

(10) A body starts with a velocity of 2 metres per second in a given direction, and has also an acceleration of 20 centimetres per second per second in a perpendicular direction. Find the velocity after 5, 10, and 20 seconds.

(11) If the velocity of a body changes in direction by 60° without changing in magnitude, find the direction and magnitude of the added component of velocity.

(12) A heavy ball thrown from a carriage window appears to a passenger to have an acceleration which is directed forward and downward at an angle of 60° to the horizon. Find the acceleration of the train.

CHAPTER VII

PROJECTILES

76. When a body is allowed to fall from rest, or is projected vertically upwards or vertically downwards, we have seen that its path is a vertical straight line. When a body is projected in a direction not vertical, a *vertical plane may be drawn which passes through the point of projection and contains the direction of the initial velocity*; and, so long as the body is moving unconstrainedly, its motion will be confined to this vertical plane; for the acceleration (g) is directed vertically downwards, and the component of velocity added in any time is therefore also vertical, and there is never any component of motion except in the plane aforesaid. As in a former chapter, the resistance of the air will be neglected, so that in most practical cases our investigations would not apply with any great degree of accuracy; they may be taken, however, as furnishing a fair approximation to the truth, and since the general nature of the case is necessarily familiar to the student, it will be useful as an illustration of the principles established above.

77. The form of the path of a projectile is shown in fig. 28. A body being projected from a point O in the direction O T will travel along a path which is tangential to O T at the point O, but which only touches O T at this one point. For, if the initial velocity be resolved into two components, one in the horizontal direction O X, the other in the vertical direction O Y, the horizontal component will remain constant, while the vertical component will constantly change. Thus, in the present case, since the vertical component of the initial velocity is upwards, and since the constant acceleration (g) is downwards, the upward velocity will become less and less, and the resultant

velocity will be more and more nearly horizontal, until, when some point A of the path has been reached, the vertical component of velocity will have been destroyed, and the motion will be entirely horizontal.

The downward acceleration, remaining unchanged, then produces a gradually increasing *downward* velocity, and throughout the rest of the motion the inclination of the body's path to the horizontal becomes continually greater.

78. Now let us consider the motion more closely. Commencing at the instant when the body is at the point A and its

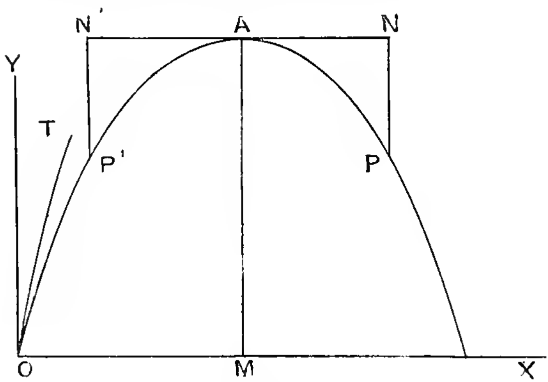


FIG. 28.

velocity is horizontal, let this horizontal velocity be called u and let the downward acceleration be g.

Then, for the horizontal component of motion,

$$\text{velocity} = u \text{ and is constant ;}$$

for the vertical component of motion,

$$\text{initial velocity} = 0 \text{ ; acceleration} = g, \text{ and is constant.}$$

Hence, to find the position (P) of the body at the end of the time t, we have

$$\text{A N} = \text{horizontal distance from A} = ut \text{ ;}$$
$$\text{N P} = \text{vertical (downward) distance from A} = \tfrac{1}{2} gt^2.$$

At the time $-t$ we have

$$A N' = \text{horizontal distance from } A = -ut.$$

$$N'P' = \text{vertical (downward) distance from } A = \tfrac{1}{2}g(-t)^2 = \tfrac{1}{2}gt^2.$$

Hence $\qquad AN^2/NP = u^2t^2/(\tfrac{1}{2}gt^2) = 2u^2/g;$

and $\qquad AN'^2/N'P' = u^2t^2/(\tfrac{1}{2}gt^2) = 2u^2/g;$

$2u^2/g$ being a constant quantity which suffers no change of value during the motion. Every point in the path, then, fulfils these conditions: (1) It lies in a certain fixed vertical plane; (2) Its *vertical* distance from A is proportional to the *square* of its *horizontal* distance from A. Thus the path of the body is a parabola with its vertex at A and its axis (A M) vertical.

At the point P the horizontal velocity $= u$; the vertical velocity = that generated by the acceleration g in the time $t = gt$ *downwards*; and hence, the magnitude of the resultant velocity at P $= \sqrt{u^2 + g^2t^2}$.

At the point P′ the horizontal velocity $= u$, as before; the vertical velocity = that which will be *destroyed* by the acceleration g in the time $t = gt$ *upwards*, or $-gt$ *downwards*; the resultant velocity at P′ having, as before, the magnitude $\sqrt{u^2 + g^2t^2}$.

It has already been shown that $P'N' = PN$, so that P and P′ are on the same level, that is, lie in the same horizontal plane; and hence, if any horizontal plane be drawn intersecting the path of a projectile in two points, the horizontal components of velocity will be the same at these two points, and the vertical components will be equal and opposite; while the resultant velocities will be equal in magnitude, and their inclinations to the horizontal equal and opposite.

Examples.

(1) From the top of a tower 122·5 metres high, a stone is projected horizontally with a velocity of 8 metres per second. Where will the stone strike the ground, and how long will it take to fall? the acceleration due to gravity being 980 centimetres per second per second.

Since the downward acceleration is independent of the horizontal motion, the time of descent will be the same as if the stone fell vertically from rest.

The height of the tower, s, = 12250 cm. ; g = 980 cm. per second per second, and the time required is therefore given by

$$s = \tfrac{1}{2} g t^2$$

or
$$12250 = \tfrac{1}{2} \cdot 980 \cdot t^2$$

whence
$$t = \sqrt{\frac{2 \times 12250}{980}} = 5 \text{ seconds.}$$

Meanwhile, the horizontal velocity has been constant, and equal to 8 metres per second, so that the horizontal displacement will be $8 \times 5 = 40$ metres ; and the stone will therefore strike the ground 40 metres from the foot of the tower.

(2) $A_0 B_0$ is a finite vertical straight line and L_0 is its middle point, while the straight line $B_0 C_0$ is horizontal, and, therefore, perpendicular to $A_0 B_0$. There are also two bodies (called A and B), whose positions at a certain instant coincide with A_0, B_0, and the point mid-way between these two bodies is called L, so that, at the same instant, L coincides with L_0. The body A, starting from A_0, falls freely along $A_0 B_0$ with acceleration g, while the body B, simultaneously starting from B_0, moves with uniform acceleration, $2g$, along $B_0 C_0$. It is required to investigate the motion of L.

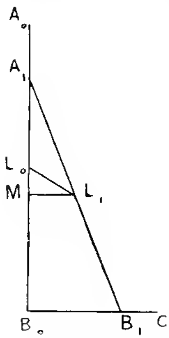


Fig. 29.

After the time t from starting, the first body will be at a point A_1 on $A_0 B_0$, $A_0 A_1$ being equal to $\tfrac{1}{2} g t^2$; and the second body will be at B_1 on $B_0 C_0$; $B_0 B_1 = \tfrac{1}{2} . 2 g t^2 = g t^2$. The point half way between the two bodies will, of course, be at L_1, the middle point of $A_1 B_1$; and thus, during the time t, the movable point L has received altogether a displacement $L_0 L_1$, which may be resolved into the two components $L_0 M$ and $M L_1$.

Now, $L_0 M = L_0 B_0 - M B_0 = \tfrac{1}{2} A_0 B_0 - \tfrac{1}{2} A_1 B_0$ (for by similar triangles $A_1 M / A_1 B_0 = A_1 L_1 / A_1 B_1 = \tfrac{1}{2}$).

Similarly, $M L_1 = \tfrac{1}{2} B_0 B_1$; and hence, during any time the vertical space described by L is half that described by A, so that L must have at each instant half the vertical velocity of A, and the change in L's vertical velocity during any time is half the corresponding change in the velocity of A. Hence, finally, L must have at each instant a downward acceleration equal to half the downward acceleration of A ; that is, equal to $\tfrac{1}{2} g$, which is constant.

From precisely similar reasoning we conclude that the hori-

zontal acceleration of L is in the direction $B_0 C_0$ and has a constant value $= . \frac{1}{2} . 2g$.

Hence, the resultant acceleration of L is constant, being made up of two constant components which are in perpendicular directions, and are equal respectively to $\frac{1}{2}g$ and g. The magnitude of the resultant is thus $\sqrt{\left(\frac{g}{2}\right)^2 + g^2} = \frac{\sqrt{5}}{2} . g$, and since the motion is from rest, the path will be a straight line, $L_0 L_1$, in the direction of the acceleration.

In the figure, $L_0 M$ and $M L_1$ may be taken to *represent* the component accelerations according to a certain scale; and $L_0 L_1$ will then represent the resultant acceleration.

Examples on Chapter VII.

$g = 980$ (C.G.S.) unless otherwise stated.

(1) Find the height of a tower if a body projected horizontally from the top, with a velocity of 9·8 metres per second, reaches the ground at a point distant 40 metres from the foot.

(2) A body is projected from a point in a horizontal plane; find its range on the plane in terms of the horizontal and vertical components of the initial velocity, and of the acceleration g.

(3) A cannon-shot is sent obliquely upwards with a velocity of 19·6 metres per second, at an inclination of 45° to the horizon. When will its velocity be entirely horizontal, and how far will it then be from its starting-point?

(4) Two bodies are projected from points in the same horizontal plane with velocities of equal magnitude, the one at an inclination of 30°, the other at an inclination of 60°; show that they will have the same range.

(5) Compare the times of flight of the two bodies in example 4.

(6) A projectile strikes the ground with a velocity of 5880 centimetres per second at an inclination of 30° to the horizon. What was the position and what the velocity of the projectile 2 seconds previously?

(7) A body is projected with a velocity of (2452·5) $\sqrt{2}$ centimetres per second from a point in a horizontal plane, its velocity

being initially inclined 45° to the plane, and it strikes the plane again after 5 seconds. Find the range of the projectile and also the acceleration due to gravity.

(8) One stone is dropped from a height, and simultaneously another stone is projected horizontally from the same point with velocity v. When the first stone has acquired a vertical velocity of magnitude v, what will be the velocity of the second stone, and what will be the distance between the two?

(9) The range of a projectile on a horizontal plane is 40 metres, and the distance is accomplished in 5 seconds. Find the horizontal and vertical components of the initial velocity.

(10) A body is projected with a velocity of 10 metres per second, and after 4 seconds it has again reached the horizontal plane through the point of projection. Find the range on this horizontal plane.

CHAPTER VIII

MATTER AND FORCE

79. We commence with some theorems and definitions.

Newton's First Law of Motion.—*Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.* The first part of this statement—that a state of rest persists so long as it is undisturbed by force—will be accepted without much difficulty. If a body which has been at rest up to a certain instant begins at that instant to move, we should feel certain that some force has acted on the body to produce this change in its condition. To illustrate the motion of a body under the action of no forces, the example usually chosen is that of a smooth stone sliding along a level sheet of ice. The stone is unable to fall downwards because the ice supports it with a force equal and opposite to its own weight, but still it is not quite free from the action of retarding forces. No matter how smooth the ice, the speed of the stone will gradually diminish until finally it comes to rest; for the air always opposes some force to the motion of the stone, and so also does the surface of the ice, which can never be *perfectly smooth.* But the smoother the ice, and the less the resistance of the air, the more slowly will the speed of the stone diminish, and we may readily admit that if all opposing forces were removed, the stone would for ever continue to move in a straight line with uniform velocity.

Again, if a stone is thrown vertically upwards it would continue for ever moving in the same direction with uniform velocity, were it not that there are forces opposed to its motion,

the chief of which is due to the earth's attraction, and is called the *weight* of the body).

The first law of motion, then, amounts to this : *When no force operates to produce a change of motion, no change of motion will take place.*

80. **Force.**—The following is Newton's definition of force: *Force is that which moves or tends to move a body, or which changes or tends to change the motion of a body.* The statement of the First Law of Motion will have prepared the student for this definition, and it is unnecessary, therefore, to offer further remarks upon it at this stage. We shall return to the subject more fully when we come to consider force as a measurable quantity. At present we must turn our attention to the measurement of matter.

81. **Quantity of Matter.**—We are accustomed to associate with the idea of matter various properties. Some of these properties depend entirely upon the *quality* or *condition* of a substance ; for example : *density*, *hardness*, *temperature*. There are other properties, again, which depend also on the *quantity* of the substance taken ; such as *volume*, *weight*, and *mass*. Any one of these latter properties may be chosen to measure the quantity of matter in a given body, and we shall now consider them in turn.

82. **Volume.**—This standard for comparing quantities of matter is familiar to us ; thus, liquids are measured in cubic centimetres or in litres, in pints or in gallons, and air is measured in litres or in cubic feet. Are we to say, then, that the *quantity of matter* in a body is measured by the *volume* which it occupies? No ; for by means of pressure all bodies can be more or less reduced in volume, so that under different conditions the *same* portion of matter may occupy *different* volumes. Let us next consider

83. **Weight.**—The weight of a body is the **force** with which it is attracted by the earth. We can form some idea of the weight of a body by holding it in the hand and noticing the degree of muscular exertion which is necessary to sustain it; but a more reliable measure is afforded by suspending the body in question from the end of a good steel spring, and observing the amount by which the spring is stretched. Thus, let A B

(fig. 30) be a spring which is firmly fixed at the end A, and suppose that when a certain body, M, is allowed to hang freely from the end B, the length of the spring is increased from A B to A B′. If the body M is now removed, the spring will regain its original length A B; and so long as its elastic properties remain unaltered, it will always take the same amount of force to elongate it to the length A B′, whether that force is due to the weight of a body, to muscular exertion, or to any other cause. All bodies, then, which, hanging freely, elongate the spring by the same amount, must be attracted downwards with equal force; that is, their weights are equal. It will be found, moreover, that no matter how the volume, or shape, or temperature of a body is altered, its weight will still remain the same; for it will always produce the same elongation of the spring.

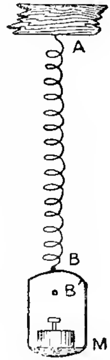


FIG. 30.

But suppose that the spring and the body M are taken to a different locality, say to the top of a high mountain. Being now more distant from the earth, the body will be attracted with a smaller force, that is, its weight will be less. On the other hand, the elastic properties of the spring will not be affected by the change of altitude, so that to increase its length from A B to A B′ still requires the same force as before. We shall therefore find, on suspending the body M, that the elongation produced is just a trifle smaller. If the apparatus could be removed to the moon, which is much smaller than the earth, the body M would be attracted by the moon with a force much smaller than its weight down here, and the elongation produced in the spring A B would be correspondingly less. If the body were taken to some distant part of space, very far from the earth or moon or other attracting bodies, it would cease to have any appreciable weight; while all the time it consists of the same identical portion of matter. Thus the weight of a body does not furnish us with an invariable and consistent measure of the quantity of matter which it contains; but, on the other hand, it will be found that the property called *mass* is perfectly suited for this purpose.

84. **Mass.**—Let A B (fig. 31) be a thin lath of wood—say about a foot or a foot and a half long, an inch or more broad, and about $\frac{1}{16}$ of an inch thick. (A straightened piece of clock-spring or of hoop-iron, or anything long and flat and springy, may be used instead.) Let the end A be clamped in a table vice, or nailed on to the vertical side of a box C, so that its length A B is horizontal while its breadth is vertical. The box being firmly pressed against the floor, let the end B of the lath be plucked with the forefinger, so as to set it vibrating horizontally: its vibrations will be more or less rapid. Now tie some small body, such as a pocket-knife, to the end B of the lath, and set it vibrating once more. It will be seen to vibrate much more slowly; but why? The *weight* of the added body can have

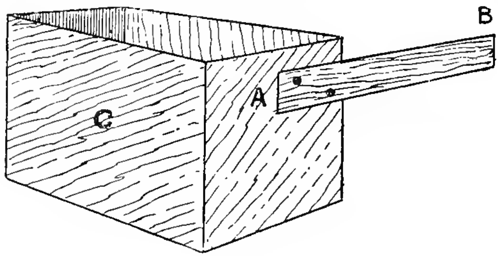


Fig. 31.

nothing to do with the effect, for the weight is a *force acting vertically downwards*, and all the motion is horizontal. But it will be evident that while the body is being moved backwards and forwards by the vibrating lath, *its velocity is constantly changing*. As it approaches the end of a swing, it slackens speed, stops, and then begins moving in the opposite direction; only to have its motion reversed again at the other limit of its swing. The slackening influence of the added body may be thus explained in general terms: To *change* the motion of matter requires the application of force; the force in this case is due to the intrinsic elastic properties of the lath, which properties are the same throughout the experiment; but when a small body is tied to the end B, there is *more matter* to be moved, and so the changes of motion are slower.

The experiment may be varied as follows: Hold the end B of the lath between the thumb and forefinger, and move it rapidly backwards and forwards; then, after tying a small body on to B, repeat the process, making the vibrations as rapid as before. It will be found that considerably more effort is required in the second case. It is *not* the *weight* of the body which calls forth this effort, for the weight is supported by the lath A B; it is simply that when more matter is added, more force is necessary to produce the same changes in its motion. We may now define mass provisionally as follows: *The mass of a body is that property in virtue of which the continued application of force is necessary to change its velocity; and two bodies are said to have equal mass when a given application of force produces in each the same change of velocity.*

85. When a train is travelling in a straight line with uniform velocity, it is subject to no force on the whole, the propelling force of the engine being then just sufficient to balance the friction of the wheels and the resistance of the air. When the propelling force of the engine exceeds the sum of these opposing forces, the speed of the train will be increasing; if the opposing forces preponderate, the speed will be decreasing.

Again, take the case of a stone attached to a piece of string and whirled around in a circle, while in contact with a smooth horizontal table. The velocity of the stone is then constantly changing *in direction*, and the force required to produce this change is supplied by the pull of the string. If the string be suddenly cut, the stone will move off along a straight line tangential to its former path.

86. When a body is allowed to fall freely *in vacuo*, the only force acting upon it is its own weight. If the motion does not take place *in vacuo* there will be an opposing force due to the resistance of the air; but when the falling body is a considerable mass of metal, and before its velocity has become very great, the resistance of the air will be very small compared with the weight of the body, and may consequently be neglected. Now, it has already been stated as an experimental fact that, when a body is falling freely, its velocity increases at a uniform rate, or, in other words, that its acceleration is constant; so the

weight of a body being a constant force directed vertically downwards, the body when moving under the sole action of this force has a constant acceleration directed vertically downwards. This is in accordance with the fundamental principle of dynamics, that *a constant force produces a constant acceleration in its own direction.*

No matter what may be the velocity of the body, the same force acting upon it will always produce the same acceleration, and the acceleration will always be in the direction of the force.

87. We have already stated this important experimental truth, that *all bodies* falling freely *at the same place* have the same acceleration. Consider, then, two bodies whose *weights at the same place* are equal : when falling freely they are subject to the *same force*, and their *velocities* change by the *same amount in the same time* ; hence, in accordance with the definition of § 84, they are said to have the *same mass* ; that is, *if the weights of two bodies* **at the same place** *are equal, their masses are equal, and vice versâ.* Remember that this is *not* a self-evident relation—mass is a quantity as distinct from weight as either is distinct from velocity, and the constancy of their relation at the same place is only known from experiment. We had obviously no right to *assume* that *in vacuo* all bodies would fall with the same acceleration, and even now that the fact is well known, we are very far from understanding why it should be so.

The same principle may be stated a little differently, as follows : *At a given place on the earth's surface equal masses are attracted downwards with equal force.* In some given place, then, let there be n bodies, each of the same mass, m, and each, therefore, having the same weight, which may be called w. These bodies considered collectively will constitute a body whose mass is $n\,m$ and whose weight is $n\,w$; and this will be true whatever number is represented by n. Whence, *in the same locality the weights of bodies are proportional to their masses.*

88. Suppose, now, that we are provided with two pieces of metal each of one ounce, with an ordinary beam-balance, with a spring-balance (fig. 30) and with an elastic lath clamped at one end (fig. 31). First place the two pieces of metal one in

each scale-pan of the beam-balance ; and the balance, if just, will remain in equilibrium, showing that each of the two bodies is acted on by the same downward force ; that is, that each has the same weight. (Indeed, it was by an experiment of this sort that the manufacturer tested their equality.)

Next suspend one of the pieces (M) from the spring-balance (fig. 30) and carefully observe the deflection. This will depend on the stiffness of the spring and on the downward force (weight) acting on the body. The other body (N) when suspended will produce the same deflection, for its *weight* is the same.

Now attach the body M to the free end (B) of the elastic lath (fig. 31), and set it vibrating horizontally. The number of vibrations executed per second depends on the *mass* of M and on the elastic properties of the lath. Then let M be replaced by N, and the number of vibrations per second will be the same as before ; for since M and N have the same weight in the same locality they must have the same mass. Next, suppose that all this apparatus—the bodies M and N, the beam-balance, the spring-balance, and the elastic lath—were removed to the summit of a high mountain and the same experiments repeated. M and N will still balance one another when placed in the scale-pans of the beam-balance ; for, though their weights will be rather less than before, they will be changed in the same proportion, and will, therefore, still be equal to one another. When M or N is suspended from the spring-balance, the elongation produced will be less than it was at the lower level, for the stiffness of the spring will have remained unaltered while the weight has been somewhat diminished. It will be found, however, that N when suspended still produces exactly the same elongation of the spring as M.

Finally, let M or N be attached to the free end of the elastic lath, which is then vibrated horizontally ; the number of vibrations per second will *not* be found to have changed, and since the stiffness of the lath has not altered, we conclude that the *mass* of a given material body is an invariable quantity, independent of the attractions of neighbouring masses.

89. We have learnt, then, that :

(1) The **volume** of a given body may vary very largely.

(2) The **weight** of a given body depends on the disposition and nature of neighbouring bodies (such as the earth), but is quite independent of the degree of compression or other physical conditions of the body itself.

(3) The **mass** of a given body is absolutely unalterable by any changes of condition or locality which we have been able to observe. It is therefore evident that the *mass* of any portion of matter is a much more fundamental and consistent measure of its *quantity* than is either its *volume* or its *weight*.

90. **The Gram.**—In the metric system, the unit of mass is the **gramme** (or in English, **gram**). It is one thousandth part of the standard kilogram prepared by Borda and preserved in the Conservatoire des Arts et Métiers in Paris, and is, roughly speaking, equal to $15\frac{1}{2}$ English grains. So long as no portion of it is worn away and no new matter added to it, the standard kilogram will always have the same mass, to whatever part of the world or of interstellar space it may be removed. A number of copies have also been prepared, each having a mass of one kilogram, and these are to be found in various places throughout the civilised world.

91. **Comparison of Masses.**—To test the equality of mass between a copy and the original by observing the motion of each under the action of known forces would be a difficult matter, nor would the method be susceptible of much accuracy, so in practice we make use of the *known principle* that, in the same locality, equal masses have equal weights, and, placing the two bodies one in each scale-pan of a balance, observe whether there is equilibrium. Though the comparison in this case is between the *weights* of the bodies, we are really preparing standards of *mass*, *not* of *force* ; for the weight of a kilogram of matter will be greater in London than in Paris, and greater in Paris than in Philadelphia ; but the standard and its copies, having all the same weight *in the same locality*, must have the same mass.

On the other hand, it requires always the *same weight* to produce the same elongation of a given steel spring with permanent elastic properties ; so that, if in London a kilogram produced a certain elongation, we should find that in Paris

something must be added to the kilogram to produce the same elongation. Thus the steel spring is essentially an instrument for comparing *weights* (*forces*), but if in the *same locality* two bodies, successively suspended, produce equal elongations of the spring, we know that their masses are equal.

92. **Atwood's Machine** (fig. 32) is an apparatus for illustrating the laws of motion, and for roughly determining the acceleration of falling bodies. A pulley, A, is supported by its axle on 'friction wheels,' B, C, which allow it to turn about a horizontal axis with the least possible friction. The pulley is grooved as shown at A′, and the string, S S′, passing over the groove in the manner indicated, is attached to the masses of metal M, M′. The other parts of the machine will be described when we come to speak of their employment.

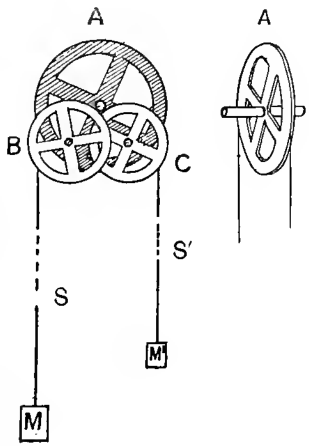


FIG. 32.

In the construction of the machine there are certain conditions which should be fulfilled as nearly as possible, though they can never be fulfilled exactly.

(1) The string and the pulley should be as light as possible, so that they may not appreciably add to the quantity of matter in motion.

(2) The string should be inextensible, so that when one of the masses (M, M′) moves downwards, the other must move upwards through exactly the same length.

(3) The string should be perfectly flexible, so that it could be bent (without stretching) to any extent by the application of the smallest possible force.

(4) The rotation of the pulley should be entirely frictionless.

To simplify the investigation we shall assume that all these

conditions are fulfilled : the quantity of matter in the pulley being so small that we may neglect the dynamical influence of its motion, the effect will be the same as if the pulley remained fixed while the string slid over it *without friction*, and thus a pull exerted at one end of the string will be transmitted without change over the pulley to the other end, the stretching force on the string being consequently the same throughout. Again, since in any time the displacements of M and M′ are equal and opposite, it follows that at each instant M and M′ have equal and opposite velocities, and equal and opposite accelerations.

93. **Convention of Signs.**—For our present purpose it will be convenient to adopt a special definition of the positive direction ; we shall call those displacements positive which take place along the length of the string in the sense of M′ A M, displacements in the sense of M A M′ being called negative. Such a convention as this involves no assumption as to matters of fact, and it is only needful to bear in mind the meaning of the expressions we employ. For example, the remarks at the end of the preceding section were made on the supposition that for each of the masses the positive direction was downwards and the negative direction upwards ; with the new convention we must say that in any given time M and M′ receive *equal* displacements, and that at each instant they have equal velocities and equal accelerations. When M is descending and M′ ascending, both are moving in the positive direction ; when M′ is descending and M ascending both are moving in the negative direction, the total moving mass being M+M′. Again, the weight (W) of the mass M is a force acting in the positive direction, while the weight (W′) of the mass M′ is a force acting in the negative direction. The force (T) with which the string is stretched is the same on each side of the pulley, and it acts in the positive direction on M, and in the negative direction on M′, so that the resultant force on the moving masses (measured in the positive direction) is W−W′.

Thus the mass in motion (M+M′) is equal to the *sum* of the masses, while the force (W−W′) acting on the moving system is equal to the *difference* of the weights.

If we remove a portion of the mass M and add it to the mass M′, or *vice versâ*, the resultant force W—W′ will be correspondingly changed, while the mass M+M′ is obviously unaltered. In this way we may examine the action of *different forces* on the *same mass*.

Or again, if we add bodies of equal mass (and therefore of equal weight) to M and to M′, the resultant force will be unaltered, while the mass in motion will be increased; and in

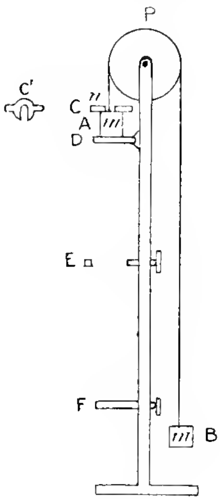


FIG. 33, *a*.

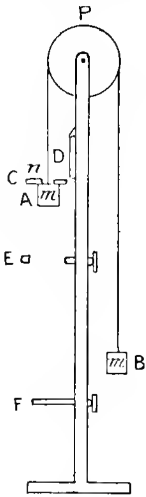


FIG. 33, *b*.

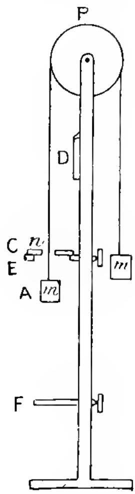


FIG. 33, *c*.

this way we may examine the action of the *same force* upon *different masses*.

94. The principle of the employment of Atwood's machine is as follows (fig. 33, *a*, *b*, *c*): two cylinders of brass, A and B, each of mass m, are attached to the ends of the string, while a third piece of brass, C, of mass n, and of the form shown at C′, is laid on the top of A (fig. 33, *a*). When this system is moving freely it is evident that the mass in motion is $(m + n) + m$ or $2m + n$, while the force acting in the direction B P A is equal to the weight of A + the weight of C − the weight of B, that

is, equal to the weight of C. Now, since in any given locality the weights of bodies are proportional to their masses, let us suppose that in the place where the experiments are made, the weight of any body is equal to k times its mass, where k is a physical quantity depending on the latitude, the height above the sea-level, &c.

We have, then, the resultant force $k\,n$, acting in a determinate direction on the aggregate mass $2m + n$.

Before the motion commences, the body A rests on the shelf D (fig. 33, *a*), and when this shelf is allowed suddenly to fall (as in figs. 33, *b*, *c*), the system begins to acquire a velocity in the direction B P C, which is here called the *positive* direction. Moreover, since the mass $2m + n$ is acted on by a constant force $k\,n$, its acceleration is uniform, and may be denoted by f.

The ring E is so arranged that A can pass freely through without touching it, while C, being too large to pass through, is left resting on the ring as in fig. 33, *c*. The mass n having thus been removed from the system, the mass in motion is $2m$, and since the resultant force is now obviously zero, the *velocity* will continue uniform, until A is suddenly stopped by coming into contact with the shelf F placed to receive it.

95. From start to finish, then, the motion of the system is as follows: Commencing from rest, the acceleration f continues to act until a certain distance has been accomplished; a velocity (v) being thus produced. The mass n is then suddenly removed by the ring E, and the remaining mass $2\,m$ continues to move with uniform velocity v until stopped by the shelf F. The observer takes note of three instants: (1) When he lets fall the shelf D; (2) when C is stopped by the ring E; (3) when A is stopped by the shelf F; (2) and (3) being each announced by the sound of a collision.

Thus we know the time (t_2) between the instants (2) and (3), during which the velocity was uniform and equal to v; and since the space s described in this time can also be directly measured, the velocity $v = s/t_2$ is known.

Again, we know the time (t_1) between the instants (1) and (2), during which the uniform acceleration f acted from rest;

and since we know the velocity v produced in this time, the acceleration $f = v/t_1$, and is also known.

96. By means of Atwood's machine the following propositions may be established :

(1) *When the resultant force is zero, the velocity remains uniform.* To prove this let us make a number of experiments in which everything remains unchanged, except the height of the shelf F. Thus, f and t being the same in each experiment, the velocity v will also be the same, and, by regulating the height of the shelf F, we may give different values to the space s in successive experiments, and may thus show that, after the removal of C, the velocity has remained constant. In practice this can only be approximately true, for the machine must necessarily be subject to some friction, which produces a retarding effect ; and the observations just described may therefore be taken as furnishing a test of the workmanship.

(2) *When the resultant force is constant, it produces a constant acceleration.* Suppose another set of experiments made, in which A, B, and C still remain unchanged, while the distance from D to E has successively different values. Then, in each experiment, the same value will be found for f, but t_1, having different values, must be measured separately in each case ; and so also must the velocity v, which is produced in the time t_1 under the acceleration f. It will thus be found that v is proportional to the time t_1 during which the force $k\,n$ has acted ; or, in other words, the force $k\,n$ while its acts, produces a constant acceleration f.

(3) *When the mass in motion is given, the acceleration is proportional to the resultant force.* (The mass in motion is $2m + n$; for A, B, and C all take part in the motion while the force $k\,n$ is acting ; and when this force has ceased to act, the uniform motion of A and B merely serves to determine the velocity already acquired.) We have seen how the acceleration may be measured when the masses of A, B, C are given, and if now a mass μ be taken from each of the bodies A and B, and the mass 2μ added to C, the total mass is unaltered, while the resultant force has changed from $k\,n$ to $k\,(n + 2\mu)$. The acceleration will be found to have changed in the same

proportion, and in this way the proposition is easily established.

(4) *When the resultant force is given, the acceleration is inversely proportional to the mass.*

If in one observation the masses of A, B, and C are m, m and n respectively, and if afterwards the mass μ be taken from each of the bodies A and B while C retains the unaltered mass n, the resultant force will still be kn, while the total mass will have changed from $2m + n$ to $2m - 2\mu + n$. The acceleration will then be found to have changed in the inverse proportion, and the truth of the proposition may thus be made evident.

97. All these theorems may be collected in the following law :

If the resultant force acting on a body is constant in direction and magnitude, the motion of the body will be subject to a uniform acceleration in the direction of the force, proportional to the magnitude of the force and inversely proportional to the mass of the body.

Let m be the mass of any given body, p the resultant force acting upon it, and f its acceleration ; then f is proportional to p/m, and we may write

$$f = \frac{p}{m} \cdot C \quad\quad\quad \textbf{(12)}$$

where C is some numerical constant. Now, when the units of length, mass, and time are fixed, the value of C will evidently depend on the unit we adopt for the measurement of forces, and as yet we have said nothing on this head.

98. **Absolute Unit of Force.**—The unit of force which most readily suggests itself, is the *weight* of a body whose mass is the unit of mass ; but a given material body would not furnish an invariable and self-contained unit of force, for we should have further to specify in what locality the body was to be placed. Again, we might take as the unit the force required to extend, by one centimetre, the length of a given spring under given conditions of temperature, &c., and such a spring would then furnish us with a definite unit of force. But the

selection in the one case of locality, or in the other case of the standard spring, remains perfectly arbitrary, and the numerical constant C may thus be made to assume any value. No new arbitrary relation, however, is necessary, for we may *define the unit force* by writing the constant C equal to unity in equation (12); and since the unit so determined depends only on the fundamental units of length, mass, and time, it is called an *absolute unit of force.*

Equation (12) being now written:

$$f = \frac{p}{m} \text{ or } p = mf \text{........................} (13)$$

it follows that, when $m = 1$ and $f = 1$, p is also $= 1$; that is, *the unit force is that which, acting on the unit of mass, produces the unit acceleration.*

99. **The Dyne.**—In the C.G.S. system the unit of force is that which, acting for one second on a free mass of one gram, produces in it a velocity of one centimetre per second, and this unit is called a *dyne.* In the British system, the unit of force is that which, acting for one second on a free mass of one pound, produces in it a velocity of one foot per second, and this unit is called a *poundal.* The word 'free,' as here employed, means that the mass is subject to no constraint, and is acted on by no force save that which we specially consider.

100. **Advantages of an Absolute System.**—It is important for the student to realise in some measure the advantages to be derived from the adoption of an 'absolute' system of units. In so-called 'gravitation' systems, the unit of force is the *weight at the place of observation* of some fixed quantity of matter—such as a gram or a pound—and so in one sense the unit is easily accessible for purposes of comparison. But in exactly the same way each observer might find it convenient to take the length of his own forearm as a standard or unit of length, and the disadvantages would be of much the same nature in the two cases. For measurements which were made in the one case of forces in different localities, or in the other case of lengths by different observers, would not be comparable with one another without first determining the relation between

the various arbitrary standards of measurement. It is much better, then, to use invariable standards of force and of length; to express the weight of a gram in dynes or of a pound in poundals; and to measure the length of an observer's fore-arm in centimetres or in feet.

In practical engineering, gravitation units are still employed, and, when great accuracy is not sought for, the disadvantages are not quite so serious; the weight of any given mass being *pretty nearly* the same in all latitudes and at all accessible elevations.

101. **Weight in Absolute Measure.**—The downward acceleration of freely falling bodies has been denoted, as is usual, by the letter g, and it has been pointed out that the acceleration g is due to a certain force (the weight of the body) acting upon the mass of the body. The unit of force has further been *defined* by the equation,

$$p = mf;$$

where m is the mass of a body, p the force acting upon it, and f the acceleration produced. Now, when a body of mass m is falling freely, p becomes equal to w (the *weight* of the body), and f becomes equal to g, so that the equation may in this case be written:

$$w = mg \quad \text{.................(14)}$$

or, in words: *The weight of a body expressed in absolute units of force is equal to the mass of the body multiplied by the local value of the acceleration due to gravity.*

A mass of one gram has a weight of g dynes, where g is expressed in centimetres per second per second; and a mass of one pound has a weight of g poundals, g being expressed in feet per second per second.

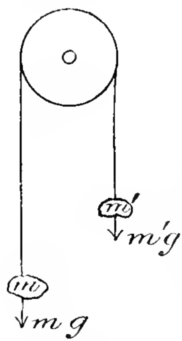


FIG. 33, *d*.

102. We may now consider, in a slightly different manner, the acceleration of the moving masses in Atwood's machine (fig. 33, *d*). Let the greater mass be m and the lesser mass m', and let the *positive* direction be reckoned always *downwards*; then, the weight of

the mass m is a force equal to $+mg$, and the weight of the mass m' is a force equal to $+m'g$. The force T with which the string is stretched is the same on each side of the pulley, and acts *upwards* on each of the bodies, that is, in the negative direction; hence, the force acting on the mass m is $mg - \mathrm{T}$, and that acting on m' is $m'g - \mathrm{T}$. Dividing each of these forces by the mass on which it acts, we obtain for the values of the respective accelerations $(mg - \mathrm{T})/m$ and $(m'g - \mathrm{T})/m'$, and since these accelerations are known to be equal and opposite,

$$\frac{m'g - \mathrm{T}}{m'} = -\frac{mg - \mathrm{T}}{m};$$

or,

$$\mathrm{T} = \frac{2m\,m'g}{m + m'} \qquad \text{(15)}$$

which follows at once on clearing of fractions and collecting terms in T.

The acceleration of the mass m has just been shown to be

$$\frac{mg - \mathrm{T}}{m};$$

that is,

$$\frac{1}{m}\left(mg - \frac{2m\,m'g}{m + m'}\right);$$

which reduces to

$$\frac{(m - m')\,g}{m + m'} \qquad \text{(16)}$$

Hence, the greater mass has a *positive* (that is, *downward*) acceleration whose magnitude is found by dividing the difference of the weights (mg and $m'g$) by the sum of the masses (m and m'). The conclusions, then, are exactly the same as in §§ 96 and 97, where the convention as to signs was quite different. In practice any convention may be adopted, provided it leads to no inconsistency or ambiguity in the expressions; but all the data of the problem must be stated consistently with the convention when once decided on, and the results of the investigation must be interpreted accordingly.

103. Two particular forms of the expression (16) may be noticed: (*a*) If $m' = 0$, the acceleration is mg/m, that is g, and m has, as might have been expected, the same acceleration as a freely falling body. (*b*) If $m = m'$, the acceleration is zero,

and the masses either remain at rest or move with uniform velocity. (See the remarks following the first proposition of § 96.)

At the same time equation (15) will assume specially simple forms: (*a*) If $m' = 0$, T becomes zero, and the string is seen to be free from stress. (*b*) If $m = m'$, $T = 2m^2 g/2m = mg$, which is equal to the weight acting at either end, and each body, since it experiences a pull equal and opposite to its own weight, will be without acceleration, for the resultant force acting upon it will be zero.

104. **Determination of** *g*.—One of the uses of Atwood's machine is to determine the value of *g*—the acceleration of falling bodies. When a body is falling freely under the action of gravity, its acceleration is so great that it cannot be conveniently observed, but in Atwood's machine the acceleration may be made as small as we please, by choosing masses m and m' which are nearly equal, so that their difference is small compared with their sum; and since by means of an ordinary balance it is easy to find m and m' in terms of the unit of mass, we can calculate the value of g in terms of the observed quantities, m, m', and the acceleration $(m - m')g/(m + m')$.

105. **Practical Details.**—So far, only those points have been noticed which bear directly on the theory of the machine; but in practice the details of working are modified by the consideration that it is easier to measure fractions of a metre or a foot than fractions of a second. Instead of setting the ring E and the shelf F (fig. 33), and measuring the times t_1 and t_2 as described in § 95, the ring E is adjusted by trial so that the body C is withdrawn from the motion after some whole number of seconds from starting; the shelf F being then similarly adjusted, so that the motion is stopped after a further whole number of seconds. The shelf D is made to fall at one beat of a seconds' clock, and the collisions with E and F are arranged to coincide with subsequent beats.

Examples.

(1) A mass of 5 grams is weighed in a spring-balance, and is found to produce a certain deflection; the balance is then removed to another locality, and a mass of 5·01 grams is found to produce

the same deflection. Compare the accelerations of falling bodies in the two localities.

Let the values of g to be compared be g_1 and g_2 respectively; then the weight of 5 grams in the first locality is $5g_1$ dynes, and the weight of 5·01 grams in the second locality is $5{\cdot}01 \,.\, g_2$ dynes; and since these forces produce equal effects, they must be equal to one another; that is,

$$5g_1 = 5{\cdot}01 \,.\, g_2;$$

or,
$$g_1 : g_2 = 5{\cdot}01 : 5.$$

(2) Two masses, each of 20 grams, are attached to the ends of a thin, flexible, inextensible string, as in Atwood's machine. What mass must be added on one side so as to produce an acceleration equal to $\frac{1}{10} \,.\, g$?

A force of $4g$ dynes would be required to produce an acceleration of $\frac{1}{10} \,.\, g$ in a mass of 40 grams, and this force is equal to the weight of 4 grams; but if 4 grams were added to one of the masses, the total mass would have become 44 grams, and we must therefore proceed as follows:

Let m be the mass required; then the total mass to be moved is $40 + m$ grams, and the resultant force is mg dynes, so that the acceleration is $mg/(40 + m)$ centimetres per second per second. The value of m is thus determined by

$$\frac{mg}{40 + m} = \frac{g}{10};$$

or,
$$10m = 40 + m;$$

whence,
$$m = 40/9 = 4 \,.\, \dot{4} \text{ grams.}$$

(3) A free mass of one pound is acted on by a force equal to the weight of two ounces; find the acceleration produced.

The weight of 2 ounces is only given as a measure of the *force*; it is to be inferred from the statement that one pound is the whole *mass* in motion.

(4) The larger mass in an Atwood's machine is 100 grams, and it moves downwards with half the acceleration it would have when falling freely; find the pull of the string, and hence deduce the value of the smaller mass.

(5) What mass would move with an acceleration of one centimetre per second per second under a force equal to the weight of one gram?

(6) A merchant buys goods near the equator, and sells them in London. Will he gain or lose by using in each place the same spring-balance?

Examples on Chapter VIII.

(1) When the unit of length is a metre, the unit of mass a kilogram, and the unit of time a minute, how many dynes will there be in the absolute unit of force?

(2) The sum of the moving masses in an Atwood's machine is equal to 20 grams. How much of this mass must be attached to either end of the cord that the acceleration may be $\frac{4}{5}$ that of a freely falling body?

(3) The pull of the string in Atwood's machine is an harmonic mean between the weights of the moving masses.

(4) The motion in Atwood's machine having continued for a time t from rest, the string is suddenly cut. What change will thus be produced in the relative acceleration of the masses?

(5) A mass of 15 grams is uniformly accelerated, and its velocity changes from 10 to 25 centimetres per second while it describes a space of one metre. What is the force producing the acceleration?

(6) A body moves from rest under the action of a constant force equal to the weight of one gram. If the distance travelled after five seconds is 75 centimetres, find the mass of the body.

(7) If the unit of mass were one kilogram, and the unit of force were the weight of one gram, what would be the unit of acceleration?

(8) A coin of mass m is placed on my hand, which is moved vertically up and down. Supposing that at some given instant the acceleration is f (measured downwards), what force will then be exerted by my hand upon the coin? What condition is necessary that the coin may remain in contact with my hand?

(9) The sum of the moving masses in an Atwood's machine being given, show that the pull of the string will be greatest when the masses on the two sides are equal.

(10) The moving masses being m and m', what must be substituted for the latter mass so as to reverse the direction without altering the magnitude of the acceleration?

CHAPTER IX

NEWTON'S SECOND LAW

106. **General Remarks.**—We have seen in previous chapters how displacements, velocities, and accelerations are compounded and resolved. The considerations there involved are entirely geometrical, and belong to the science of pure motion. The displacement of a body which moves during a given interval of time with a given velocity, and all such-like relations, are independent of any material properties, and even of the absolute measurement of time. Nothing was said in the first seven chapters as to how time was measured, nor was any *definition* given of successive *equal* intervals of time. If the time were measured by a clock whose hands had a continuous forward movement, it would not modify the statement of any purely kinematical laws if the clock went fast or slow or varied erratically, provided all the phenomena were referred to the same clock. We should then *define* equal intervals of time as those in which equal angles were traced out by the minute-hand; and we should say that a velocity was constant in magnitude when the distance travelled during any interval of time bore a constant ratio to the length of the arc simultaneously described by the end of the minute-hand.

107. **Absolute Measurement of Time.**—But when we come to consider the motion of material bodies, the phenomena cannot be directly inferred without recourse to observation and experiment; and, moreover, in estimating the movements of material bodies, the absolute measurement of time becomes fundamentally important. Thus, the First Law of Motion tells us that a body unacted on by force travels equal distances in equal times, and we are necessarily led to consider how equal intervals

of time are determined. Practically, the most accurate measurements of time are based on observations of the earth's rotation about its axis ; but there is no essential reason why the rotation of some other planet should not be adopted as a standard of uniformity, and the results would not necessarily be the same in the two cases ; for there are certain causes (notably the tides) which tend to make our planet rotate more slowly, so that, after the lapse of many centuries, the period of its rotation may have appreciably increased, and the extent of such retarding influences would be different for different planets. If we could accurately observe the rotation of some celestial body which appeared to be free from all dynamical influences tending to alter its period, we should give to it the preference as a standard of uniform motion, and the result would then be the same as if we measured the lapse of time in accordance with Newton's First Law of Motion, equal intervals of time being defined as follows : *If a material body be subject to the action of no forces, equal intervals of time are those in which the body receives equal displacements.*

108. **Fixity of Direction.**—It will be seen that this definition, so far as it relates to speed, is merely a paraphrase of Newton's First Law ; but the same law tells us that a body unacted on by force moves in a constant direction, and it thus supplies a *definition* of a direction fixed in space. If the directions of enormously distant bodies, such as the 'fixed' stars, be taken as standards, we shall be led to sensibly the same results, and here, as before, we are depending on dynamical principles ; for the stars, being material bodies, cannot, without some sufficient cause, acquire velocities so enormous as to be perceptible at this distance.

109. **Force.**—Again, the measurement of acceleration depends on that of direction, of length, and of time, while a given material body furnishes an invariable standard of mass, so that by means of equation (12) or (13) we are enabled to compare and to measure *forces*, and either of these equations furnishes a *definition* of force as a measurable quantity, the latter serving also to define the absolute unit of force.

The acceleration of a material body is thus accepted as a

criterion of the constancy of the force by which it is actuated: a body falling through the air is found to have an acceleration which continually diminishes; that is, its velocity *increases less quickly* as time goes on, and we therefore conclude that the force which acts on the body is not constant, the variation being chiefly due to the resistance of the air, which resistance increases with the speed of the falling body. The acceleration of any material body is determined by its mass and the direction and magnitude of the resultant force on the body, independently of the actual velocity.

110. In an absolute system of units

$$\textbf{force} = \textbf{mass} \times \textbf{acceleration},$$

and the force acting upon a body at any instant may be measured as follows (fig. 34): Let O P represent, according to a certain scale (τ), the velocity of the body at the given instant, and let O Q represent the velocity after the lapse of a very short time t; then, since during the time t the velocity changes from (=) O P to (=) O Q, P Q represents in direction and magnitude the velocity added during this very brief time, and P Q/t therefore represents the *acceleration* of the body according to the scale τ. This acceleration, multiplied by the mass, gives in absolute measure the *resultant of all the forces* acting on the body.

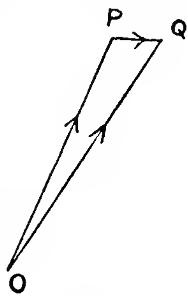


FIG. 34.

If the direction of the force—and consequently, also, of the acceleration—is inclined to that of the velocity, this latter direction will obviously change, and the path will be curved, as in the general case of projectiles (Chapter VII). The matter may now be considered from a slightly different point of view by introducing the quantity called

111. **Momentum.**—*The momentum of a body is the product of its mass by its velocity*, and is evidently a *directed* quantity, having, in fact, the same direction as the velocity itself. A momentum may be represented on a diagram in the same way

as any other directed quantity, the scale of the diagram being the quotient of an interval of time (τ) divided by a mass (μ), as follows: If O P (fig. 35) be drawn to represent the momentum M, then a body of mass μ, which uniformly described the path O P in the time τ, would have the momentum M.

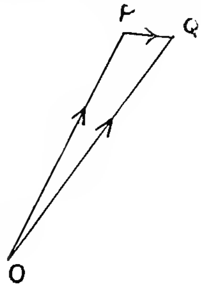


Fig. 35.

Again, since the momentum of a given mass is proportional to its velocity and in the same direction, it is evident that momenta will follow the same laws of composition and resolution as velocities. Now, let O P represent according to the scale τ/μ the momentum of a body at a given instant, and let O Q represent its momentum after a very short interval of time t; then P Q represents the momentum added during the time t, and we have:

Initial momentum = mass × initial velocity,
Final momentum = mass × final velocity;

whence it immediately follows that the added component of momentum is equal to the mass multiplied by the added component of velocity. Hence,

rate of change of momentum
= mass × rate of change of velocity
= mass × acceleration,

which is equal to the resultant of all the forces acting on the body; and the fundamental dynamical equation may now be written:

force = rate of change of momentum
= mass × acceleration..................... **(17)**

the sign = being here taken to imply identity of direction as well as equality of magnitude.

Starting, now, with the equation $p = mf$, let the force p continue uniformly for the time t. Then, during this time, the motion of the body is subject to a constant acceleration f, and

v, the change of velocity produced in the time t, is equal to $f t$ in direction and magnitude; while $m v$, the change of momentum, $= m f t = p t$. The relation

$$p t = m v \quad \text{......} \quad (18)$$

which has just been established, may be thus expressed in words: *If a body free to move be acted on by a constant force, the momentum produced will be measured by the product of the force into the time during which it acts, and will be in the same direction as the force.*

If the body were initially at rest, $p t$ would be its final momentum; otherwise the final momentum must be found by compounding $p t$ with the initial momentum. It must further be remarked that unless p is the resultant of *all* the forces acting on the mass m, p/m will not be the complete measure of the acceleration. If there are other forces in operation, whether arising from gravity or from any other cause, p/m will be the *component* of acceleration due to the force p, and similarly each of the other forces (such as q) will produce an acceleration (q/m) in its own direction. Combining all these components in accordance with the principles already laid down, we obtain the resultant acceleration *at the instant considered.* This will be true even if the forces are not constant; p, q, &c., being then taken to represent their instantaneous values.

112. **Quantity of Motion.**—It will be seen that the differences between dynamical and purely kinematical conceptions are essentially due to the introduction of the quantity called mass. The velocity of a point or of a geometrical figure is a purely abstract conception, but the velocity of a material body (relative to some other material body) is a physical reality. The momentum of a body, that is its velocity multiplied by its mass, may be taken to measure the *quantity of motion*, so that in this sense there is more motion in a river than in a small stream which flows with the same rapidity, and no motion at all in an immaterial point or geometrical figure, whatever its velocity.

113. **Newton's Second Law of Motion.**—It was in this sense that the word 'motion' was understood by Newton, his

Second Law of Motion being enunciated as follows: *Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.* It has already been shown that change of momentum is equal to the product of the resultant force by the time during which it acts (18), and that the force itself is measured by the rate of change of momentum (17). Thus, the Second Law of Motion when translated into modern language becomes: *Rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.* When *absolute* units are employed, the word *proportional* in this statement is to be replaced by *equal.*

Examples on Chapter IX.

(1) Find the momentum of a cannon-shot whose mass is 8 kilograms, and which moves with a velocity of 30 metres per second.

(2) Two bodies are travelling with velocities of 10 and 15 centimetres per second respectively, and their momenta are equal. Compare the masses of the bodies.

(3) If a given force acting for a given time on a given body produces in it a velocity of v centimetres per second, what velocity will be produced by a force k times as great acting m times long on a body of n times the mass?

(4) A mass m is subject to the action of a constant force p. What momentum will have been acquired during the time t?

(5) A body whose mass is one kilogram is initially travelling northward with a velocity of 6 metres per second, along a perfectly smooth horizontal plane. For how long must it be acted upon by a westerly force of 5,000 dynes that the direction of its motion may become north-westerly? State the direction and magnitude of the change of momentum produced by the force.

(6) The momentum of a train is 6×10^{11} C.G.S. units, and the train is brought to rest in ten seconds by applying the brakes. Find the retarding force due to friction.

(7) If the acceleration of falling bodies were 980 (C.G.S.), what mass would have a weight equal to the retarding force in example 6?

CHAPTER X

COPLANAR FORCES—EQUILIBRIUM

114. **Parallelogram of Forces.**—The laws of the composition and resolution of forces are easily deduced from this principle: that when a number of forces act simultaneously on a body, each produces the same acceleration as if it had acted separately, and that the resultant force is the product of the mass and the resultant acceleration. Let two forces p and q act on a body of mass m; the components of acceleration due to these forces are p/m and q/m respectively, and may be represented, as in fig. 36, by two straight lines O P, O Q; their resultant being represented by the diagonal O R of the parallelogram O P R Q. But the component forces are respectively $m \times$ the acceleration (=) O P and $m \times$ the acceleration (=) O Q; the resultant force being $m \times$ the acceleration (=) O R. Thus, according to a new scale, O P, O Q may be taken to represent the component forces p, q in direction and magnitude, and O R will then represent the resultant force. Hence the proposition: *If two forces be represented in direction and magnitude by two straight lines drawn* **from** *a point, and if a parallelogram be constructed having these two straight lines for adjacent sides, the resultant of the forces will be represented in direction and magnitude by*

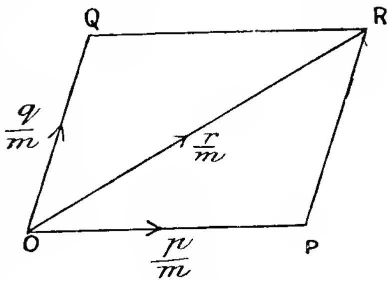


FIG. 36.

the diagonal of the parallelogram which is drawn **from** *that point.*

115. Hence follows the **polygon of forces,** and in general it will be seen that forces follow the same laws of composition and resolution as other directed quantities.

116. From all this it is evident that if a number of forces act simultaneously on a body during any given time, *the same change of momentum* will be produced as if the forces had acted separately, each for an equal time; for the resultant change of momentum is obtained by compounding the changes due to the several forces, and is independent of the order of composition.

117. When forces are thus represented on a diagram, the *scale* will be the product of two intervals of time τ_1, τ_2, divided by a mass μ; thus, if a force represented by A B were to act for a time τ_1 on a mass μ, it would generate in it a certain velocity; and this velocity, continuing uniformly for the time τ_2, would produce a displacement equal to A B.

Examples.

(1) A mass of 10 grams is acted on by a horizontal force of 5,000 dynes as well as by its own weight. What is the value of the resultant force and of the acceleration?

(2) Show by means of a diagram that when two forces of given magnitude act on a body, their resultant is diminished by increasing the angle between their directions.

(3) Being given the magnitudes of two forces, and also of their resultant, employ the triangle of forces to find the angles between their directions.

(4) A body is moving in a north-easterly direction with a momentum of 5,000 C.G.S. units, and a force of 200 dynes in a north-westerly direction then acts upon it for 18·75 seconds. What is the magnitude of the final momentum?

118. **Point of Application.**—So far our attention has been confined to movements of translation and to forces tending to produce such movements; any effect which the impressed forces may have in producing rotation has not as yet been taken into account, an omission which would only be strictly

justifiable if the mass of the body were collected at a single point; that is, compressed to an infinitely small volume. Practically, the mass must be of finite extent, and it is evident that forces such as P and Q (fig. 37), whose points of application are A and B, will tend to rotate the body in the direction of the hands of a watch, as seen from our point of view.

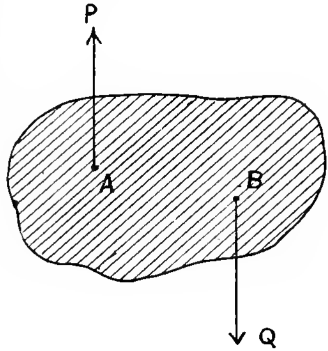


FIG. 37.

A force is completely determined when we know

(1) Its direction.

(2) Its magnitude.

(3) Its point of application.

A force of given direction and magnitude has the same effect on the *translation* of a body whatever its point of application, but a similar relation will not be found to hold when *rotation* comes to be considered. When any number of forces act upon a body, which are known in direction and magnitude, their resultant can be determined in direction and magnitude by employing the polygon of forces, but to find the point of application of the resultant requires us to know the points of application of the several forces.

119. The first proposition to be established is this: *When a force given in direction and magnitude acts upon a rigid body, the effect will be unaltered by transferring the point of application of the force to any other point in its line of action.* (It must be remembered that a force has *at each instant* a definite value, and when the *effect* of a force is spoken of, it has reference to the accelerations at that instant, which are due to the action of the force, whether these accelerations affect the translation or rotation of the body.)

Let M be a rigid body (fig. 38) and P an impressed force whose point of application is A. Take any point B in the line of action of P, and at B let a force Q act, which is equal and

oppositely directed to P; then the forces P and Q, acting simultaneously, will be without effect; for it is evident that they do not tend to produce a motion of translati on, and it is equally evident that they will not produce rotation. Moreover, since the body is rigid, it will not suffer any change of figure from the application of the forces; so that P and Q completely annul one another's action. If, now, the point of application of the force P be transferred to any other point A′ in its line of action, P and Q will be in equilibrium as before, and hence the effect of a force is not changed by transferring its point of application to any other point in its line of action; and the magnitude and *line of action* of a force will completely determine its effect on a rigid body. Observe that 'line of action' *includes* direction—the line of action of P is B A, while that of Q is A B.

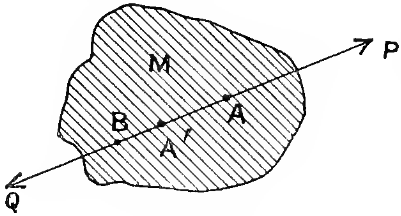


FIG. 38.

120. **Rigid Bodies.**—It is necessary to specify that the body is *rigid*, for otherwise its figure would be changed by the application of force, and the above demonstration would cease to be strictly applicable. Practically, no body is really rigid in this sense, every substance yielding to some extent even under the smallest stress; but the rigidity of many bodies is so great that their deformation *by the forces considered* is inappreciable, and is consequently neglected. It is in this sense that the term 'rigid' is used in dynamics.

121. **Resultant of Non-parallel Coplanar Forces** acting on a rigid body.—Let P and Q be two forces, A and B their points of application, and let their lines of action be produced to meet in C. Each of the forces may now be supposed to act at C, and if it happen—as in the present case—that C falls outside the body, we must suppose this point joined to the body by *rigid immaterial* connections, and we may then find a single force R, which, acting at C under these imaginary con-

ditions, would produce the same effect as the actual forces applied at A and B.

If P and Q be represented by C M, C N, according to any convenient scale, then the diagonal C L of the parallelogram C M L N will represent their resultant R, according to the same scale. C L is also the line of action of the resultant, for it is evident that if each of two forces acts at a given point, their resultant is a force acting at the same point.

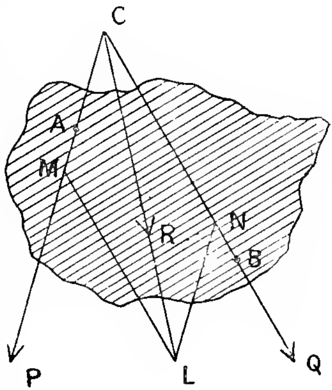


FIG. 39.

Similarly, R may be compounded with a third force S; the magnitude and line of action of the resultant of R and S being found as before; and so, generally, we may find the line of action and magnitude of the resultant of any number of coplanar forces.

122. **Resultant of Two Like Parallel Forces**; that is, forces in the *same direction.* The first step in the previous section

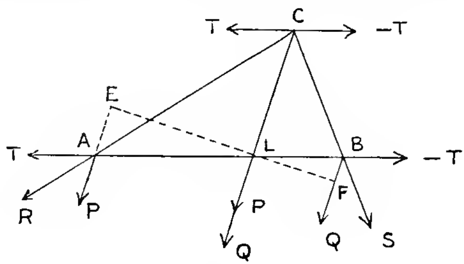


FIG. 40.

was to find the intersection of the lines of action of two forces; if, then, the forces are parallel, the method will not be applicable without modification; we may, however, proceed as follows: Let P and Q be two parallel forces acting in

the same sense at the points A, B, and join A B. Let a force of any magnitude act at A in the direction B A and be called + T, while a force of equal magnitude acts at B in the direction A B and is called − T.

These two forces annul one another's effect and have therefore no influence on the resultant. Compounding P and T at A, we obtain a resultant R acting at A, while Q and − T at B have a resultant S acting at B.

Produce the lines of action of R and S to meet at C; then the resultant of R and S passes through C.

Now R, acting at C, may be resolved into two components having the same direction and magnitude as P and T at A; and S, acting at C, may be resolved into two forces having the same direction and magnitude as Q and − T at B. The forces T and — T annul one another, and we are left with a force P + Q acting at C in the same direction as P at A or Q at B. Let the line of action of the resultant P + Q meet A B in L; then, in the triangle C L B, C L, L B, are respectively in the directions of the forces P and − T, while C B is in the direction of their resultant S. Hence, it follows from the triangle of forces that

$$CL/LB = Q/(-T);$$

and, similarly, from the triangle C L A,

$$CL/LA = P/T.$$

Dividing the first of these equations by the second we obtain

$$LA/LB = -Q/P \text{ or } P \cdot LA + Q \cdot LB = 0 \quad \ldots\ldots\ldots (19)$$

If the straight line E L F be drawn perpendicular to C L (or in any other direction) and meeting A P, B Q, produced if necessary, in E and F respectively, then E and F may be taken as the points of application of the forces, and we shall find as before

$$LE/LF = -Q/P \text{ or } P \cdot LE + Q \cdot LF = 0 \ldots\ldots\ldots (19a)$$

Since the given forces act in the same sense, P and Q have the same sign, and L E, L F having, therefore, contrary signs, are drawn in opposite directions; that is, L must lie between

E and F. Hence, *When two parallel forces act in the same sense on a rigid body, their resultant acts in the same direction, and is equal in magnitude to their sum; while its line of action lies between those of the component forces in their own plane, and divides the distance between them in the inverse ratio of the forces.*

123. **Resultant of Two Unlike Parallel Forces.**—Let the parallel forces P and Q act in contrary senses at the points E and F (fig. 41); these points being so chosen in the lines of action of the forces that E F is perpendicular to their directions. Now add the forces + T at E and − T at F, which act in opposite directions along the straight line E F; and let R be the resultant of P and T at E, S the resultant of Q and − T at F. In general, the lines of action of R and S will

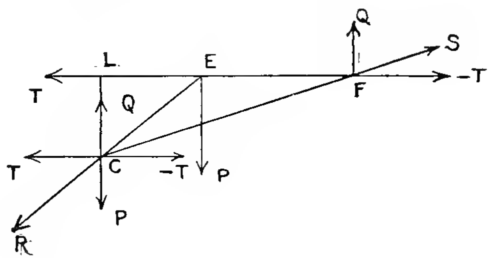


FIG. 41.

meet at some point C, at which each of these forces may then be supposed to act. R may now be resolved into two forces, P and T at C, and S into two forces, Q and − T acting at the same point. Of these four components, T and − T annul one another, and we are left with a force P + Q acting in the direction in which P and Q are measured; but since P and Q have contrary signs, P + Q is *numerically* equal to their difference, and is in the direction of the greater, as is also directly evident from the figure.

To determine the line of action L C of the resultant P + Q, we have, as in the previous case,

$$\mathrm{CL}/\mathrm{LF} = \mathrm{Q}/(-\mathrm{T}),$$
$$\mathrm{CL}/\mathrm{LE} = \mathrm{P}/\mathrm{T};$$

whence, as before,

$$LE/LF = -Q/P, \text{ or } P \cdot LE + Q \cdot LF = 0 \ldots\ldots(19a)$$

Since the given forces act in opposite directions, Q/P is a negative quantity, and $-Q/P$ is consequently positive; hence, LE and LF, having the same sign, are drawn in the same direction, and LC does *not* therefore lie between the lines of action of P and Q. Thus, *When two forces act in opposite directions on a rigid body their resultant is numerically equal to their difference and acts in the direction of the greater; while its line of action lies in the same plane as those of the given forces, and externally divides the distance between them in the inverse ratio of their magnitudes.* It will be observed that the equation (19*a*) is always applicable to the case of two parallel forces, due account being taken of the *signs* of the quantities involved, and in each case *the resultant is nearer to the greater force.*

124. **Couple.**—If P and Q act in contrary directions (fig. 41), and are very nearly equal in magnitude, then LE/LF is very nearly unity and $(LE - LF)/LF$ is very small; that is, EF is very small compared with LF; or, in other words, LF is very great compared with EF. At the same time $P + Q$ is very small, so that we have for the resultant a *very small force* acting at a *very great distance.* The effect of such a force on the translational motion of the body would be very small, but we shall see that it has, in general, a finite rotational tendency. In the limiting case, *when the two forces are equal in magnitude and opposite in direction, they are said to constitute a couple*, and it will presently be shown that the effect of a couple is purely rotational.

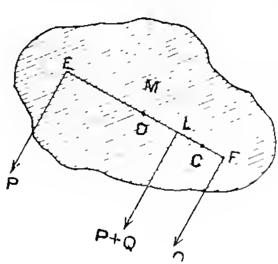


FIG. 42.

125. **Moment of a Force about a Point.**—Let P and Q be two parallel forces acting on a rigid body M (fig. 42), the lines of action of P and Q, and of their resultant $P + Q$, being as indicated in the figure. Draw ELF perpendicular to the direction of these forces and cutting their

three lines of action in E, F and L. Then, if a point D of the body, between E and L, were fixed, the resultant force would tend to turn the body in the direction of rotation of the hands of a watch; while, if the point C of the body, between F and L, were fixed, the tendency would be to turn the body in the contrary direction. If the point L were fixed (or, indeed, any point in the line of action of the resultant), there would be no tendency to turn the body about it in either direction; in other words, the tendency of Q to rotate the body about L in the direction of clock-hands would be just balanced by the tendency of P to rotate it in the contrary direction.

Thus, *A body will have no tendency to turn about a fixed point under the action of two parallel forces when, but not unless, the resultant of the forces passes through the fixed point;* and in this case we have (19*a*),

$$P \cdot LE + Q \cdot LF = 0,$$

whether the forces are like or unlike.

Now, *the product of a force into its perpendicular distance from a point is called the moment of the force about the point*, and moments in a given plane will have the same or contrary signs according as they tend to produce rotation in the same or in contrary directions. Thus, the moment of P about L is $P \cdot LE$, while the moment of Q about L is $Q \cdot LF$; and the above equation shows that these equilibrating moments are equal and opposite.

126. It will now be shown more generally, that *if two coplanar forces act on a body of which one point in the plane of the forces is fixed, and if the moments of the forces about the fixed point are equal and opposite, the body will be in equilibrium.* For, let P and Q (fig. 43) be two coplanar forces, whose moments, $P \cdot OM$ and $Q \cdot ON$, about the fixed point O are equal and opposite, so that, numerically,

$$P \cdot OM = Q \cdot ON \text{ and } OM/ON = Q/P \ldots\ldots\ldots\ldots (a)$$

Let the lines of action of P and Q meet at C; join O C and draw O B parallel to M C and O A parallel to N C, as in the figure. Then O A C B is a parallelogram, and the triangle O A C is equal to the triangle C B O; that is,

$$\tfrac{1}{2} OM \cdot CA = \tfrac{1}{2} ON \cdot CB, \text{ or } OM/ON = CB/CA;$$

whence, remembering equation (a) above, $CB/CA = Q/P$. If, then, the force P be represented by C A, according to a certain scale, Q will be represented by C B according to the same scale, and the resultant of P and Q will be represented by C O, and since this resultant passes through the fixed point O, it has no tendency to rotate the body in either direction.

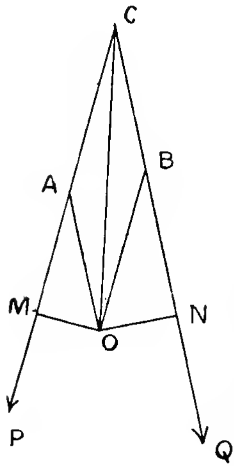


FIG. 43.

Similarly, if P were replaced by another force S, whose moment $S \cdot OK$ about the point O was equal to, and in the same sense as, $P \cdot OM$, the combined action of the forces S and Q would produce no rotation about O. Thus, *If two coplanar forces have equal moments about a given point in their plane, their tendencies to produce rotation about this point are equal*, and it is therefore evident that *the tendency of a force to produce rotation about a point is measured by its moment about that point.*

127. **Representation of Moments.**—Let A B represent a given force P in line of action and magnitude; then the moment of this force about the point O is represented by the product $AB \cdot OM$, where O M is the perpendicular from O on A B; that is, by twice the area of the triangle O A B (for we know by elementary geometry that the area of a triangle is half the area of the rectangle on the same base and of the same altitude). When an area is described by travel-

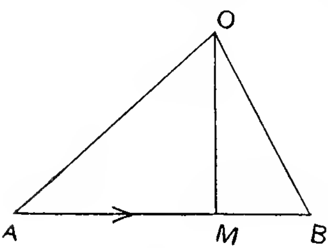


FIG. 44.

ling round in the direction of clock-hands, it is usually reckoned *negative*; areas described in the contrary direction being then taken as positive; and it is evident from the figure that the direction A B of the force P is sufficient to determine in which direction the area of the triangle O A B shall be described; for the direction in which the force P tends to produce rotation about O is evidently that of the letters O A B taken in this order. Thus, *If a force is represented in line of action and in magnitude by a given finite straight line, and if a triangle be constructed whose base is this straight line and whose vertex is at a given point; then twice the area of the triangle, taken with its proper sign, will represent the moment of the force about the point.* We may now establish the following theorem:

128. *If any number of coplanar forces act on a rigid body, the (algebraic) sum of their moments about any point in their plane is equal to the moment of their resultant about the same point.*

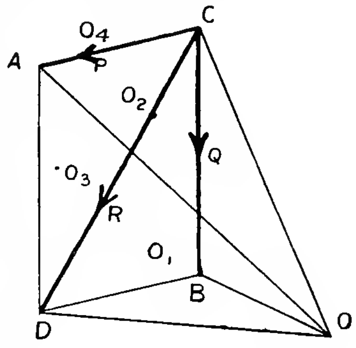


FIG. 45.

Let C A, C B represent two given forces, P and Q (fig. 45), in line of action and magnitude, and let C D represent their resultant R. Take any point O in the plane of the parallelogram A C B D; then the sum of the moments of P and Q about O shall be equal to the moment of R about O.

On our diagram these three moments are represented by the areas 2O C A, 2O C B, 2O C D, respectively, each of which, being measured in the positive direction, will be a positive quantity. Now, the triangles O C B, C A D, and O A D, all stand on equal bases (C B and A D), and the altitude of O A D is equal to the sum of the altitudes of the other two; hence, the area

$$\text{O C B} + \text{C A D} = \text{O A D},$$

and therefore, $\text{O C A} + \text{O C B} + \text{C A D} = \text{O C A} + \text{O A D}$

$$= \text{the quadrilateral O C A D} = \text{O C D} + \text{C A D}.$$

Omitting C A D from each side of this equation and multiplying by 2 we obtain

$$2OCA + 2OCB = 2OCD,$$

which proves the truth of the proposition in the case here considered ; and in every case it will be found that a similar result is obtained. If the point O were taken at O_1, within the angle formed by C D, C B, the moment of Q about O_1 would be negative, that of P would still be positive, while the moment of R would be positive and *numerically* equal to the difference of the moments of P and Q. For a point O_2 on C D the moment of P would be positive, that of Q numerically equal but negative, and that of R zero. Taking the point at O_3 between the lines of action of P and R, the negative moment due to Q would be greater than the positive moment due to P, and the moment of R would be negative. Around O_4 both P and Q have moments which are in the negative direction of rotation, and the moment of R about O_4 is negative and numerically equal to their sum.

The enunciation is true, then, in every case where the forces are two in number, and it may be very easily extended. For let $P_1, P_2, P_3, \ldots P_n$ be any number of forces acting in a plane, and let two of the forces, P_1, P_2, be replaced by their resultant R_2. Then the moment of R_2 about a given point O in the plane of the forces is equal to the sum of the moments of P_1 and P_2, and the sum of the moments about O due to the whole system will not, therefore, be altered on replacing P_1, P_2 by their resultant R_2. Similarly, R_2, P_3 may be replaced by their resultant R_3, and so on until finally we come to the resultant of all the forces R_n, whose moment about the point O is thus seen to be equal to the sum of the moments due to $P_1, P_2, P_3, \ldots P_n$. Thus the proposition is completely established.

129. **Moment of a Couple.**—Let P and $-$ P be the forces composing a couple, l the perpendicular distance between their lines of action. Take any point O in the plane of the couple and draw O B A perpendicular to the forces, as in fig. 46; then the moment of $+$ P about $O = -P \cdot OA$ (being in

the negative direction of rotation), and the moment of $-P = +P \cdot OB$; hence the resultant moment about $O = -P \cdot OA + P \cdot OB = -P \cdot BA = -Pl$; the *sign* attributed to the moment indicating in which direction it tends to produce rotation.

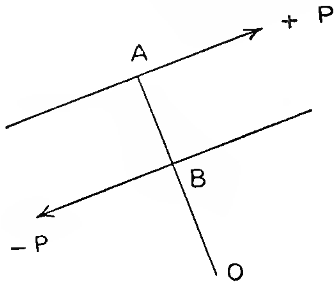


FIG. 46.

This result depends only on the forces composing the couple; it is independent of the position of the point in the plane of the couple; and we have thus proved the following proposition: *The moment of a couple is the same about every point in its own plane, and is measured by the product of either of the equal forces with the perpendicular distance between them, taken with its proper sign.*

130. **Coplanar Couples.**—If there are n couples acting in the same plane, they may be resolved into $2n$ separate forces, and the sum of the moments of these forces about a given point in their plane will be equal to the moment of their resultant. It should be remembered that the resultant of a couple, or of a number of coplanar couples, may be regarded as an infinitely small force acting at an infinitely great distance (see § 124). At the same time the moment of each couple about the given point will be equal to the sum of the moments of its component forces. It will thus be evident that *The moment due to the combined action of the* n *coplanar couples is equal to the sum of the moments of the couples taken separately, and is the same about every point in their plane.* We may also proceed as follows:

131. **Theorem.**—*Two couples in a plane will balance one another if their moments are equal and opposite.* Let P, – P, Q, – Q, be the forces composing the couples, and let their lines of action intersect at the points A, B, C, D (fig. 47). Draw A M perpendicular to B C, and A N perpendicular to D C; then the moments of the couples about any point in their plane

are respectively $-P \cdot AM$ and $-Q \cdot AN$, and these are numerically equal to one another.

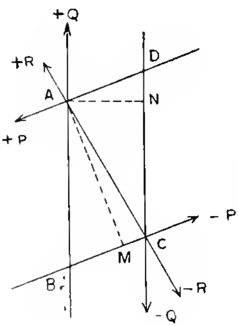


FIG. 47.

Again, $BC \cdot AM$ = area of parallelogram $ABCD = DC \cdot AN$, and hence, $BC/DC = AN/AM = P/Q$ (numerically). If, then, BC be taken to represent the force $-P$, DC will represent the force $-Q$ according to the same scale; and the other forces, P and Q, will be represented by DA and BA respectively. Thus the resultant R of P and Q $(=)$ CA, and the resultant $-R$ of $-P$ and $-Q$ $(=)$ AC; these two equal and opposite forces acting along the same straight line and annulling one another's effect; so that, on the whole, there is no moment about any point in the plane. The proposition is therefore established.

132. **Equivalent Couples.**—It immediately follows that *a couple may be replaced by any other couple in the same plane whose moment is equal and in the same sense;* for *either* of these couples would exactly neutralise the effect of a coplanar couple whose moment was equal and opposite.

133. Suppose, now, that there are n couples which act in one plane and whose moments are respectively $M_1, M_2, \ldots M_n$, each being understood with its proper sign. Choose a straight line AB of any length l, lying in the plane of the couples, and let forces numerically equal to M_1/l act at A and B respectively in opposite directions perpendicular to AB (fig. 48). If M_1 tends in the positive direction of rotation, these forces will be applied as in the figure; in the contrary case they will be as indicated by the dotted arrow-heads.

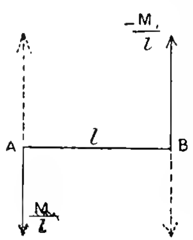


FIG. 48.

Taking due account, then, of *signs*, the couple M_1 may be replaced by the pair of forces M_1/l and $-M_1/l$, the arm of whose couple is l; and when a similar

substitution has been made for each of the remaining couples, the whole system will have been reduced to a force $(M_1 + M_2 + \ldots + M_n)/l$ acting at A, and an equal and opposite force acting at B. The moment of this couple,

$$\frac{M_1 + M_2 + \ldots + M_n}{l} \cdot l = M_1 + M_2 + \ldots + M_n,$$

which is the algebraic sum of the moments due to the separate couples.

134. **Couples in Parallel Planes.**—*Two couples in parallel planes, acting on the same rigid body, will balance one another if their moments are equal and opposite.* This is an extension of § 131, and is easily established as follows: Choose any

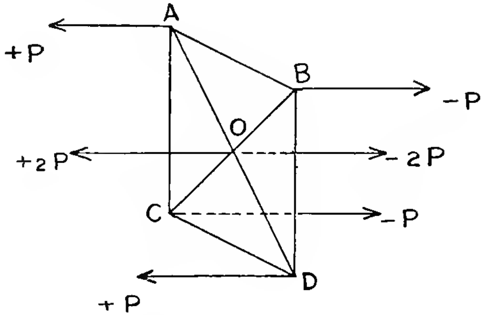


FIG. 49.

straight line A B in the plane of the first couple, and an equal and parallel straight line C D in the plane of the second couple (fig. 49). Then the first couple may be replaced by two forces, + P and − P, acting perpendicularly to A B at A and B respectively; a similar substitution may be made for the second couple, and since its moment is equal and opposite to that of the first, the force applied at C will be − P, while that at D will be + P.

Now, A B, C D, being by construction equal and parallel, A B D C is a parallelogram, and its diagonals A D, B C bisect one another at O. The resultant, then, of the equal and similarly directed forces at A and D is a force 2P acting at O,

and the resultant of the forces at B and C is an equal and opposite force $-2P$ also acting at O. The two given couples are therefore in equilibrium.

135. It follows immediately that *A couple may be replaced by any couple of equal moment acting in a parallel plane and tending in the same direction of rotation;* and also that *If a body be acted upon by any number of couples lying in parallel planes it will experience a moment equal to the algebraic sum of the moments due to the separate couples.*

136. **Theorem.**—*Any system of coplanar forces may in general be replaced by a single force acting through a given point in their plane, together with a couple.*

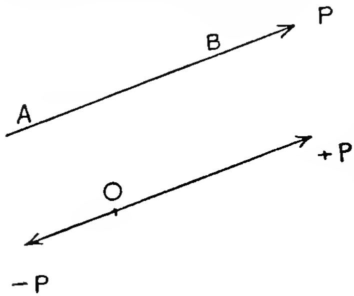


FIG. 50.

Let O be the given point (fig. 50), and P, acting along A B, one of the forces of the system. Then, if the forces $+P$ and $-P$ be made to act at O they will not in any way modify the result. But the force P along A B and the force $-P$ at O together constitute a couple, and this couple, together with the force $+P$ at O, is equivalent to the original force P. When a similar substitution has been made for each of the forces of the system, we shall have a number of forces acting through O, together with a number of couples. The resultant of the former is a single force through O, and the resultant of the latter is a couple. Thus the proposition is established; but it remains to consider some particular cases.

(1) The force acting through O and the couple may both vanish, the system being in equilibrium.

(2) The couple alone may vanish, there being then a single resultant acting through O.

(3) The force acting through O alone may vanish, the system reducing to a couple.

The forces, then, may have: (1) No resultant whatever,

(2) A single resultant through O. (3) A resultant couple. (4) A resultant force through O together with a couple. The fourth case may be further simplified by placing the couple with its components parallel to the single force through O; there will then be three parallel forces whose algebraic sum is *not* zero, and which are, therefore, reducible to a single resultant *not* passing through O.

137. Having now considered all possible cases, we see that the following proposition has been proved: *A system of coplanar forces not in equilibrium can always be reduced to a single resultant or a couple*, and it has already been pointed out in § 124 that a couple is merely the particular case of an infinitely small force acting at an infinitely great distance.

138. It follows as a corollary to § 136, that the single force through O will be independent of the position chosen for this point in the plane of the forces, being the same in direction and magnitude as if each of the original forces had acted at O. Thus, if we are given any system of coplanar forces, we may find the *direction and magnitude* of their resultant by using the polygon construction; and should the resultant so found be zero, we should know that the system was either in equilibrium or was reducible to a couple.

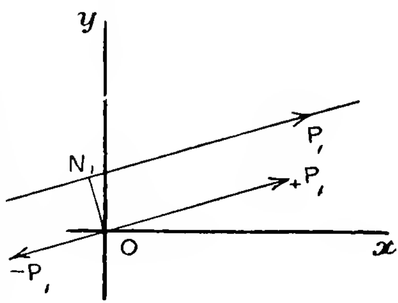


FIG. 51.

139. **General Conditions for the Equilibrium of any Coplanar Forces** acting on a rigid body.—Let P_1, P_2, . . . be the forces, and in their plane let any two straight lines Ox, Oy be chosen which are perpendicular to one another. The direction and magnitude of the resultant may be found as follows: Resolve the force P_1 into two components, X_1, Y_1, measured in the directions Ox, Oy, respectively, and let each of the remaining forces be similarly treated. Then, for the component of

force acting in the direction Ox, we have the algebraic sum $X_1 + X_2 + \ldots$, the component in the direction Oy being $Y_1 + Y_2 + \ldots$; the magnitude of the resultant force R is determined by

$$R^2 = (X_1 + X_2 + \ldots)^2 + (Y_1 + Y_2 + \ldots)^2,$$

and it is evident that R cannot be zero unless *each* of the components $X_1 + X_2 + \ldots$ and $Y_1 + Y_2 + \ldots$ is zero, for since each of the terms $(X_1 + X_2 + \ldots)^2$ and $(Y_1 + Y_2 + \ldots)^2$ is the square of a real quantity, neither of them can be negative. Since Ox, Oy are any two perpendicular axes in the plane of the forces, the first two conditions for equilibrium are that **the algebraic sum of the components resolved in each of two perpendicular directions must vanish** (1, 2). When these conditions are fulfilled we know that the magnitude of the resultant is zero, but this in itself is not sufficient for equilibrium, for it may be that the system reduces to a couple; in this case, however, the algebraic sum of the moments will be the same about every point in the plane, and the only *further* condition required for equilibrium will be that **the algebraic sum of the moments about some point in the plane must vanish** (3). Thus (1, 2) and (3) are the necessary and sufficient conditions for the equilibrium of any coplanar forces.

140. **Analytical Expressions.**—Let ON_1 be the perpendicular from O on the line of action of the force P (fig. 51); then by § 136, P_1 may be replaced by an equal and similarly directed force P_1 acting through O, together with a couple $P_1 \cdot ON_1$. The force P_1 at O may now be resolved into two components: $X_1 = P_1 \cos \alpha_1$, and $Y_1 = P_1 \sin \alpha_1$; where α_1 is the angle between Ox and the direction of P_1. We must have, then, using similar notation for the other forces,

$$\left.\begin{aligned} P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots &= 0 \\ P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \ldots &= 0 \\ P \cdot ON_1 + P_2 \cdot ON_2 + \ldots &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(20)$$

each of the component forces, and each of the component momenta, being taken with its proper sign.

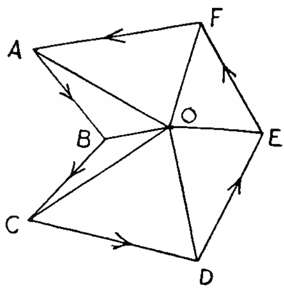
141. When a system of coplanar forces is not in equilibrium,

the same principle may be applied to determine the resultant. For the components $P_1 \cos \alpha_1 + \ldots$ and $P_2 \cos \alpha_2 + \ldots$ may be replaced by a single force through O, and this may again be compounded with the couple $P_1 \quad ON_1 + \ldots$ to form a single resultant.

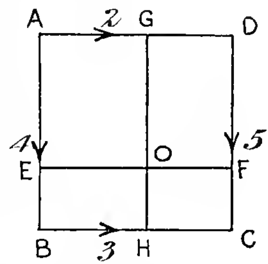
Examples.

(1) Forces are represented in line of action and magnitude by the sides of a closed polygon taken in order. Show that they are altogether equivalent to a couple which is represented by twice the area of the polygon.

Let the forces be represented by A B, B C, C D, D E, E F, F A, and let any point O be taken within the polygon A B C D E F. Then the system of forces will, by § 138, reduce to a couple, and the moment of this couple is simply the algebraic sum of the moments of the given forces about any point (such as O) in their plane. Now, the moment about O of the force represented by A B is represented by twice the area of the triangle O A B, and, when each of the forces has been similarly considered, it is evident that the sum of their moments about O is (=) twice the area of the polygon A B C D E F, and tends to produce rotation in the direction indicated by this order of lettering. Had the point O been chosen *outside* the polygon, the result would, of course, have been the same; the area then appearing as the *numerical difference* of two areas described in contrary directions, and affected, therefore, with contrary signs.



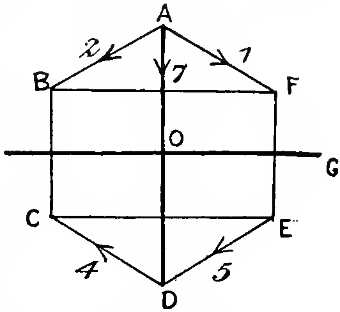
(2) A B C D is a square. A force of 4 dynes acts along A B, 5 dynes along D C, 3 dynes along B C, and 2 dynes along A D. Find the resultant force.



The resultant of the first two forces is 9 dynes acting along G H, which is parallel to A B or D C; A G/G D being equal to $\frac{5}{4}$. Similarly, we can find the resul-

tant (along E F) of the other two forces, and the final resultant will pass through O, the intersection of E F and G H, its direction and magnitude deing determinable by the parallelogram construction.

(3) A B C D E F is a regular hexagon, and is acted on by the following forces: 2 dynes along A B, 1 dyne along A F, 4 along D C, 5 along E D, and 7 along A D. Find the resultant.



Let D A be taken for one axis, the other being O G, the perpendicular to A D through its middle point O, and let the positive directions along these axes be O A and O G, the positive direction of rotation being, as usual, opposed to the motion of clock-hands. The signs to be attributed to the various forces and moments are best inferred from an inspection of the diagram. As regards the *magnitudes* of angles, it is evident that A B, A F, E D, D C, all make angles of 60° with O A and of 30° with O G; also, if the length of each side of the hexagon be called $2a$, the perpendiculars on the sides from the point O will each be equal to $\sqrt{3}\,a$.

For the components of force we shall have :

Force along	X-component (dynes).	Y-component (dynes).	Moment about O (dyne-cm.).
A B	$-2 \cdot \cos 30°$	$-2 \cdot \cos 60°$	$+2 \cdot \sqrt{3}\,a$
D C	$-4 \cdot \cos 30°$	$+4 \cdot \cos 60°$	$-4 \cdot \sqrt{3}\,a$
E D	$-5 \cdot \cos 30°$	$-5 \cdot \cos 60°$	$-5 \cdot \sqrt{3}\,a$
A F	$+1 \cdot \cos 30°$	$-1 \cdot \cos 60°$	$-1 \cdot \sqrt{3}\,a$
A D	0	-7	0

(The only formula used in the above is $Q = P \cos \theta$, where Q is the resolved part of the force P in a direction making an angle θ with the direction of the force; the corresponding formula for displacements being equation (10) of § 57.)

Remembering that $\cos 30° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$,

the component along O G $= -10\frac{\sqrt{3}}{2} = -5\sqrt{3}$ dynes;

the component along O A = $-7 - 4 \times \frac{1}{2} = -9$ dynes ; and
the moment about O = $-8\sqrt{3} \cdot a$ (dyne-cm.).

The resultant force R can now be determined in direction and magnitude, and, further, it is known that the line of action of R must be at such a distance from O that the moment about this point is $-8\sqrt{3} \cdot a$.

Examples on Chapter X.

(1) Forces of 3,000 and 4,000 dynes act at a point in directions at right angles to one another. Find the magnitude of their resultant.

(2) If three forces acting at a point are in equilibrium, show that no one of them can be greater in magnitude than the sum of the other two.

(3) If two forces of equal magnitude act at a point, what must be the angle between their directions that the resultant may have the same magnitude as either component?

(4) A B C D is a parallelogram. Show that the forces represented in line of action and magnitude by A C, B D, C B and D A are in equilibrium.

(5) If a body acted upon by a number of forces moves in a straight line with uniform velocity, what condition must the forces satisfy?

(6) The side of a square is of length a, and forces p, q, r, s respectively act along the sides of the square taken in order. What is their moment about the centre of the square?

(7) If a number of coplanar forces have the same moment about two points in their own plane, show that the straight line joining these two points is parallel to the resultant of the forces; and hence show that, if the forces have the same moment about three points not in the same straight line, they may be reduced to a couple.

(8) Like parallel forces proportional to 1, 2, 3, 4, 5, and 6 respectively act at the successive corners of a regular hexagon. Determine the line of action of their resultant.

(9) A C B is an isosceles triangle, right-angled at C, and a is the length of either side. Find the line of action and

magnitude of a force whose moments about A, B, C, are M, M, M′, respectively.

(10) Two forces act at a point in directions perpendicular to one another; one of the forces is 100 poundals and the resultant is 101 poundals. What is the magnitude of the other force?

(11) The ends of a string 7 metres in length are attached to two points in the same horizontal line and distant 5 metres from one another. A mass of 100 grams is attached at a point 3 metres from one end of the string and is allowed to hang freely. Find the pull exerted by each part of the string.

(12) Show that if three forces are in equilibrium their lines of action must lie in one plane.

(13) A pole 12 metres long is supported at its two ends in a horizontal position. From what point of the pole must a heavy body be suspended that one of the supports may bear $\frac{4}{5}$ of its weight?

(14) Forces proportional to 5, 12 and 13 act at a point and are in equilibrium. Find the angles between their directions.

(15) Masses of 5 and 7 pounds respectively are attached to the ends of a weightless rod two feet in length. At what point must the rod be supported that it may be able to rest horizontally?

CHAPTER XI

MACHINES

142. **Smoothness.**—*Two bodies are said to be smooth when the forces which they exert upon one another at their surface of contact are at each point perpendicular to that surface.* This definition will be better understood from the following considerations. If a body, A, be fixed while another body, B, can slide upon it, then, at the point where the bodies are in contact, the force R exerted by A upon B is called the *reaction* of A, and may in general be resolved into two components : one of these, N, is perpendicular to the surfaces of A and B at their point of contact and prevents B from penetrating A ; the other component, T, being tangential to the surfaces and *opposing* any tendency of B to slide over A. The perpendicular component of the reaction is called the *resistance* of A, and the tangential component is called the *friction*. Now, the force N cannot exceed a certain limit without causing serious deformation or crushing of the bodies, and this limit will depend, amongst other things, on the strength of the material composing them. There is also a limit which the force T cannot exceed, and this limit depends chiefly on the degree of roughness of the surfaces, and on the force N with which they are pressed together. If this limit is exceeded by some external force tending to make B slide, then sliding will take place. It will further be evident that the rougher the surfaces of the bodies in contact, the greater (*cœteris paribus*) will be the limiting value of the

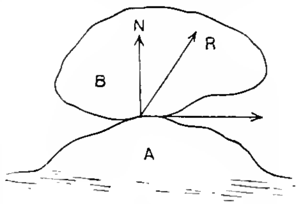


FIG. 52.

friction T, while, if the surfaces are comparatively smooth, the friction will have a small limiting value, and the application of a small tangential force will produce sliding. If the bodies could be perfectly smooth, the friction would always be zero, and the whole reaction between the bodies would be the force N perpendicular to their surfaces.

It need hardly be said that no surface to be found in nature fulfils this condition of perfect smoothness; but in many cases the amount of friction is small enough to be neglected, without causing our results to differ very much from the truth. Moreover, it is from the consideration of ideally simple cases that we arrive at a knowledge of principles, by the application of which complex phenomena may be resolved into more simple constituents.

143. **Equilibrium on a Smooth Inclined Plane.**—When a body rests on a smooth inclined plane, fewer conditions are necessary for its equilibrium than if it were unconstrained; for, in the latter case, the resultant of its weight and of the other forces impressed upon it must be zero; while, in the former case, the resultant of the impressed forces need not be zero, but may have any magnitude, provided it acts perpendicularly towards the plane. The plane will then oppose this resultant with an equal and opposite force, so that the resultant of *all* the forces acting on the body is zero, as before.

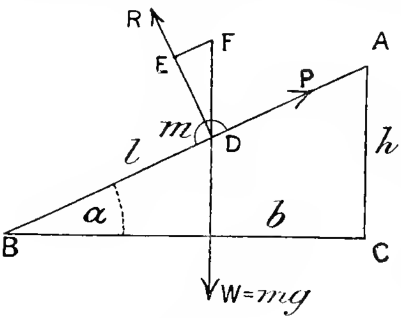


FIG. 53.

Through the position occupied by the body m, let a vertical plane be drawn, perpendicular to the given inclined plane, and cutting it in A B (fig. 53); also draw A C vertical and B C horizontal in the manner indicated, and let B A ($= l$) be called the *length* of the plane, C A ($= h$) the *height*, and B C ($= b$) the *base*.

The forces acting on the body are :

(1) Its weight W ($= mg$), directed vertically downwards.

(2) The resistance R, acting perpendicularly to the inclined plane.

(3) Some force P which is applied to preserve equilibrium. First of all, suppose P to act in a direction parallel to B A ; take any point E in the line of action of R, and draw E F parallel to B A, meeting the line of action of W in F. Now, W, R and P, being in equilibrium and respectively in the directions of F D, D E, E F, may be represented by these straight lines in direction and magnitude ; and since the angle E F D = F D A = D A C, the right-angled triangles F E D, A C B are similar, and E F/F D = h/l; D E/F D = b/l; that is,

$$P/W = h/l, \text{ and } R/W = b/l \ldots\ldots\ldots\ldots\ldots(\mathbf{21})$$

144. It is evident that the force P, which must act along the length of the plane to preserve equilibrium, is never greater than W, and neither is the resistance R. We may suppose the force P to be due to the pull of a string which is attached at one end to the body m, and at the other end to the point A of the inclined plane. If, then, the plane were horizontal, h would be zero and b would coincide with l, and we see from (21) that the string would not be stretched with any force ; the resistance R of the plane being equal and opposite to W. If the plane were now gradually tilted, the string would become stretched with a gradually increasing force, while the resistance R exerted by the plane would diminish. Finally, when the plane had become vertical, h would coincide with l, and b would be zero ; the body no longer exerting any force against the plane, and its weight being balanced by the equal and opposite pull of the string.

If α be the inclination C B A of the plane B A, we have $b/l = \cos\alpha$ and $h/l = \sin\alpha$, so that equation (21) may be written :

$$P = W \sin\alpha\ ;\ R = W \cos\alpha \ldots\ldots\ldots\ldots\ldots(\mathbf{21a})$$

145. Next, let the equilibrium of the body be maintained by a horizontal force, P (fig. 54), the lines of action of the weight W and o the resistance R cutting B C in E and F respectively. Then we shall have the angle D F E equal to the

complement of A B C = B A C, so that the right-angled triangles D F E, B A C are similar; and hence,

$$h : l : b = \mathrm{EF} : \mathrm{FD} : \mathrm{DE} = \mathrm{P} : \mathrm{R} : \mathrm{W};$$

or,

$$\mathrm{P}/\mathrm{W} = h/b\ ;\ \mathrm{R}/\mathrm{W} = l/b \quad\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{22})$$

which may also be written:

$$\mathrm{P} = \mathrm{W} \tan \alpha\ ;\ \mathrm{R} = \mathrm{W} \sec \alpha \quad\ldots\ldots\ldots\ldots\ldots(\mathbf{22}a)$$

146. Thus the horizontal force P may be either greater or less than W, according to the inclination of the plane; but the resistance R is never less than W. Supposing the force P to act always horizontally, let us trace the changes in its magnitude and in that of R as the inclination α is varied. Starting with the plane horizontal, h will be zero and b/l will be unity, no horizontal force being required to support the body, and the weight being balanced by the equal and opposite resistance of the plane. As the plane is tilted, P gradually increases and so does R, and when the inclination of 45° is reached, P will be equal to W, h/b being unity. When the plane is nearly vertical, h/b and l/b will both be very great, each becoming infinite in the limiting case when A B is quite vertical. This simply means that no *horizontal* force, however great, could support a heavy body against a *perfectly smooth* vertical plane; for, throughout these four sections, we have assumed an entire absence of friction.

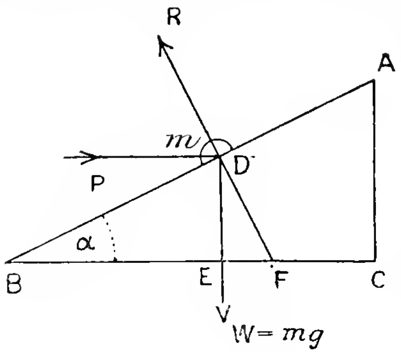


FIG. 54.

147. **Mechanical Advantage.**—When, with the aid of some mechanical contrivance, a force P is made to balance or to overcome some given force W, the given force is frequently termed the 'weight' (being usually the weight of some material body), and the force required to balance or to overcome the

weight is called the 'power.' In some machines the application of a small 'power' is sufficient to balance a considerable 'weight,' and the machine is then said to have considerable *mechanical advantage*—a quantity which is measured by the ratio W/P of the weight to the power. It may be remarked that this hardly accords with the usual acceptation of the term 'advantage'; for, when the power was equal to the weight, we should say that no advantage of power had been gained by the use of the machine, while the 'mechanical advantage' would not then be zero, but unity; and even when a machine is working at a disadvantage, the ratio W/P, though less than unity, is still essentially positive.

For the mechanical advantage in § 143, we have $W/P = l/h = 1/(\sin \alpha)$; while in § 145, $W/P = b/h = 1/(\tan \alpha)$.

148. **The Wedge.**—*A wedge is a solid body, bounded in part by two plane surfaces which are not parallel*; it is often employed for driving asunder two bodies, or two parts of the same body. Practically, friction is often the principal resistance to be overcome; but in order to simplify matters, and to show more clearly the essential nature of the wedge as a mechanical power, it will be assumed for the present that the wedge is perfectly smooth (§ 142), so that the reactions between it and the bodies with which it is in contact are entirely perpendicular to its faces.

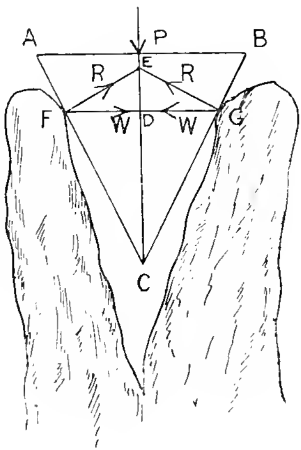


FIG. 55.

Fig. 55 represents a section of a wedge and of a body which it cleaves, the plane of the section being perpendicular to both faces, A C, B C, of the wedge (that is, perpendicular to its edge). F and G are the points where the wedge is in contact with the resisting body, the reactions (R) at these

points being along F E, G E, which are perpendicular to the faces ; and if F and G are equidistant from C, C E will bisect F G perpendicularly at D. The power is a force P, tending to drive the wedge downward ; and the weight (W) may be regarded as the component of R along F G—that is, the force which must be exerted between F and G to keep them in their present position.

NOTE.—It might appear at first sight, perhaps, that since there are *two* reactions, each equal in magnitude to R, the 'weight' should be considered as *twice* the component of either along F G, but it is evident that if the wedge were replaced by an elastic spring between F and G, the force compressing this spring would be the force acting at *either end* of it ; and it may further be noticed that the force P calls into play an equal and opposite force (the resistance of the ground) which acts on the lower end of the log. In all cases when equilibrium exists amongst a number of forces acting on a system of material bodies, each force tends to produce amongst the bodies some *mutual* displacement, which is prevented by the action of the other forces ; but this question will be more fully considered later on.

For equilibrium, P must be equal and opposite to the resultant of R and R ; and if these latter forces are represented by F E and G E, P will be represented by 2E D, the 'weight,' W, being represented by F D. Now, the sides E D, F D, F E of the triangle F D E are respectively perpendicular to the sides A P, C P, C A of the triangle C P A ; hence, these two triangles are equiangular, and therefore similar. Thus,

$$W/P = FD/2ED = CP/2AP = CP/AB \quad \ldots\ldots \textbf{(23)}$$

from which it is evident that the smaller the angle (A C B) of the wedge, the greater is the mechanical advantage.

149. **The Pulley.**—The essential nature of a pulley has been indicated in the description of Atwood's machine (§ 92). A 'smooth' pulley is one which turns without friction in its bearings, while, at the same time, the cord passing over it is perfectly flexible ; this being the case, the pull of the cord will be the same on each side of the pulley—which is sometimes expressed by saying that *a force transmitted by a flexible string passing over a smooth pulley is transmitted without change.*

When the power and the weight are applied at opposite

ends of a flexible cord passing over a fixed pulley, they must be equal in magnitude to produce equilibrium, and the mechanical advantage is consequently unity.

150. **A Single Movable Pulley** (fig. 56).—Let a flexible string pass round the pulley A, which supports a mass m, the weight mg of this mass being called W; and let one end of the string be fixed while the other is sustained by a force P, called the power. First, suppose the pulley so supported that both parts of the string are vertical. Now, the string being stretched with a force P on one side of the pulley, must be stretched with an equal force on the other side; so that, owing to the pull of the string, the pulley experiences two forces of magnitude P, both acting vertically upwards, as indicated by the arrow-heads, and having, therefore, an upward resultant 2P. The other forces acting on the pulley are its own weight and the weight, W, of the mass which it supports; and if the pulley be light enough for its weight to be neglected in comparison with W, we have this latter force balanced by 2P. So far, then, as magnitude is concerned, we have

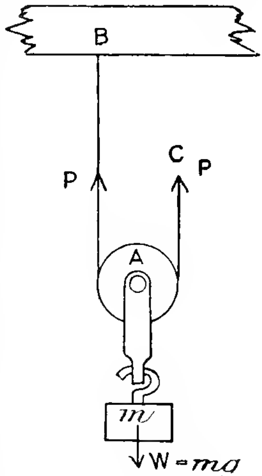


FIG. 56.

$$W = 2P \quad \text{.........................} \quad \mathbf{(24)}$$

It will be remarked that the weight W exerts a pull on the two portions of the string, each portion bearing half of the entire stress.

151. Next, let the two parts of the string, E B, D H, be inclined to the vertical (fig. 57); and, since the resultant of their equal pulls balances a vertical downward force, their inclinations to the vertical must be equal. Produce B E, H D to meet in F, take F G along F B equal to F H along F H, and complete the parallelogram G F H K, which is evidently a rhombus, its

diagonals G H, F K bisecting one another perpendicularly at O. The forces exerted on the pulley by the two parts of the cord may be represented by F G, F H respectively, and their resultant by

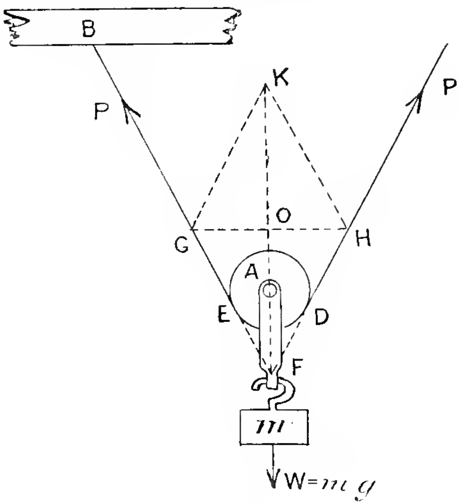


FIG. 57.

F K. The weight W, being equal and opposite to this resultant, will be represented by K F = 2 . O F. Now, if the angle O F G or O F H = α, cos α will be equal to F O/F G ; and F O = F G cos α, or F K = 2F G cos α. Numerically, then, we have

$$W = 2P \cos \alpha \qquad \text{(24}a\text{)}$$

that is, the *mechanical advantage* is 2 cos α, or *twice the cosine of half the angle between the two parts of the cord.*

When both parts of the cord are vertical, the angle between them is zero, and W/P = 2 cos 0° = 2 ; the result previously obtained (§ 150).

152. **The First System of Pulleys.**—Let the force P be applied at one end of a cord which passes over a smooth, movable pulley A (fig. 58), the other end being fastened to a fixed beam M N. Neglecting its own weight, the pulley A will

be in equilibrium when acted on by a force equal to 2 P directed vertically downwards; but, instead of allowing the weight to act directly on this pulley, let a cord be attached which passes round a second movable pulley B and is stretched with a force equal to 2P. The pull of this cord is the same on either side of the pulley B, so that, for equilibrium, B must be acted upon by a downward force equal to 4P. This force, again, may be supplied by the pull of a cord which passes round a third movable pulley C; this last being therefore able to support a downward force equal to 8P. Finally, the heavy body hung from the fourth movable pulley, D, exerts a downward force W = 16P.

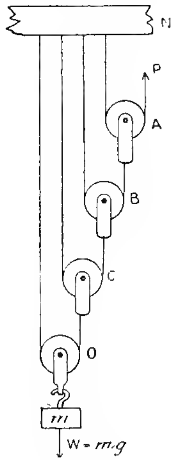


FIG. 58

In general it is evident that, in a system of pulleys arranged on the above plan, the mechanical advantage is doubled by each succeeding pulley; and with n movable pulleys we shall have

$$W/P = 2^n \ldots\ldots\ldots\ldots\ldots (25)$$

153. So far the *weights* of the movable pulleys have been neglected; suppose that they are W_A, W_B, W_C, W_D, respectively, and that the stretching forces of the strings passing round them are P, T_B, T_C, T_D. Consider the equilibrium of A: the pull T_B, together with the weight W_A, is balanced by 2P, or $T_B = 2P - W_A$; similarly, $2T_B$ balances $T_C + W_B$,

or

$$T_C = 2T_B - W_B = 2(2P - W_A) - W_B = 4P - 2W_A - W_B;$$

and, proceeding in precisely the same way, we shall find

$$W = 16P - 8W_A - 4W_B - 2W_C - W_D;$$

and, in the general case, where there are n pulleys of weights, $W_1, W_2, \ldots W_n$,

$$W = 2^n P - 2^{n-1} W_1 - \ldots - 2W_{n-1} - W_n \ldots (25a$$

154. **The Third System of Pulleys.**—It was found in the first system of n movable pulleys whose weight was small enough to be neglected, that $W = 2^n P$; and since P acts upwards and W downwards, the total force exerted on the sustaining beam M N is $W - P$ downwards, that is $(2^n - 1)P$. Suppose, now, that the whole system is turned upside down, all the forces being likewise reversed (so as to *retain their former relation* to the configuration of the system); the 'power' P pulling the free end of the cord downwards (fig. 59), and the force $(2^n - 1)P$ (which has now become the weight) acting also downwards, while the upward force required to support the pulley D is $2^n P$. For the mechanical advantage we have

$$W/P = 2^n - 1 \quad \text{..........(26)}$$

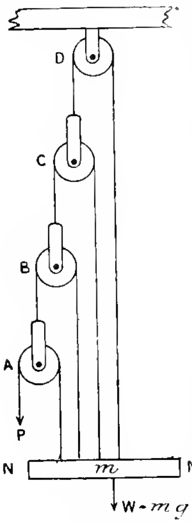


FIG. 59.

If the weights of the pulleys are too considerable to be neglected, we must remember that they still act downwards, so that their relation to the other forces is reversed. Thus we must write $-W_A$ for W_A, &c., in the equations of the previous section, and for *the upward force supporting the pulley* D, we obtain $16P + 8W_A + 4W_B + 2W_C + W_D$; and this is balanced by the downward force (now called W) acting on M N, together with the weights, W_A, W_B, W_C, W_D, of the four pulleys and the downward force P; that is,

$$W + W_A + W_B + W_C + W_D + P = 16P + 8W_A + 4W_B + 2W_C + W_D;$$

or,

$$W = (16 - 1)P + (8 - 1)W_A + (4 - 1)W_B + (2 - 1)W_C.$$

It is worth while to trace this relation more directly. Using, as before, W_A, &c., for the weights of the pulleys; P, T_B, &c., for the stretching forces of the strings, it is evident that T_B supports the weight of the pulley A together with twice the pull P of the first string, or $T_B = 2P + W_A$. Similarly, $T_C = 2T_B + W_B =$

$4P + 2W_A + W_B$, and, finally, $T_D = 8P + 4W_A + 2W_B + W_C$. Now, the body M N experiences an upward pull from each of these strings, and for equilibrium it is necessary that

$$W = P + T_B + T_C + T_D = 15P + 7W_B + 3W_C + W_D,$$

which agrees with the former result.

In general, where there are n pulleys arranged according to the third system, we shall have

$$W = (2^n - 1)P + (2^{n-1} - 1)W_1 + \ldots + (2-1)W_{n-1} \ldots\textbf{(26}a\textbf{)}$$

the weight W_n of the last pulley being absent from the equation, since its weight is supported by the fixed beam, while the other $n - 1$ pulleys are movable.

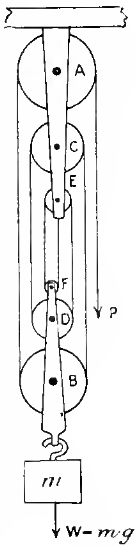


Fig. 60.

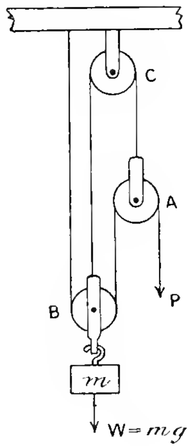


Fig. 61.

155. **The Second System of Pulleys.**—This consists of a number of pulleys, A, C, E, turning in a single fixed block (fig. 60), with an equal number of pulleys, B, D, F, turning in a single movable block. A cord passes in succession over A, B, C, D, E, F; its end being fastened to the fixed block, and its other end, which is free, being pulled by the 'power' P. Thus, throughout the whole length of the cord the stretching force is the same, and if there are n pulleys in each block, the lower block is evidently sustained by $2n$ times the pull P. For equilibrium the sustaining force must be equal in magnitude to the 'weight' W, together with the weight W′ of the lower block and its pulleys; that is,

$$W = 2nP - W' \ldots\ldots\ldots\ldots\ldots\ldots\textbf{(27}\,a\textbf{)}$$

If W′ is negligible compared with W, we may write:

$$W/P = 2n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\textbf{(27)}$$

156. As a further example, take the system of pulleys represented in fig. 61, A and B being movable while C is fixed; and let the weights of A and B be W_A and W_B respectively. The forces P and W are applied in the manner indicated; the pull, T, of the string passing over C supports the weight W_A, together with twice the pull, P, of the other string; that is, $T = 2P + W_A$. The forces acting upward on the pulley B are P, P and T, and these must balance the downward forces W and W_B; hence,

$$W + W_B = 2P + T = 2P + 2P + W_A,$$

or
$$W = 4P + W_A - W_B.$$

157. When dealing with systems of smooth pulleys the following general remarks should be borne in mind. The forces applied to the system are transmitted by means of cords, each of which has throughout its length the same stretching force; and the problem really consists in finding the stretching force of each separate cord. That of one cord is usually equal to the 'power' P, and by considering the equilibrium of each movable pulley, those of the other cords are easily deduced. Thus we arrive at the value of W required for equilibrium.

158. **Equilibrium of a Rigid Body constrained to Turn about a Fixed Axis**, and incapable of any other motion.—As an example of a body thus constrained, any one of the wheels of a clock may be mentioned, and the only forces which need be considered are such as tend to produce rotation about the fixed axis; the influence of each force being measured by *its moment about the axis*, a quantity to be presently defined.

Let A B be the axis about which the body can turn (fig. 62), and let P be one of the forces acting on the body (but not necessarily in the plane of the paper). Draw D E, the common perpendicular to A B and to the line of action E M of P, meeting the former in D and the latter in E. Through E draw E L parallel to A B, and E K perpendicular both to E D and E L. Then, since E L, E M, E K are all perpendicular to D E, they all lie in the plane through E perpendicular to D E, and P may be resolved into two components, T and N, acting along E L

and E K respectively. The product N • D E is then called the moment of the force P about the axis A B, and measures the tendency of P to produce rotation about this axis ; for it will be shown that two forces acting on the constrained body will produce equilibrium if their moments are equal and opposite. In the first place, it is evident that the component T, parallel to A B, has no tendency to produce rotation around this line, so that P will have the same effect as N, and may be replaced by this latter force. Moreover, N · D E has already been defined as the moment of N about the point D, tending to rotate the arm D E in the plane of N and D, that is, to rotate the arm

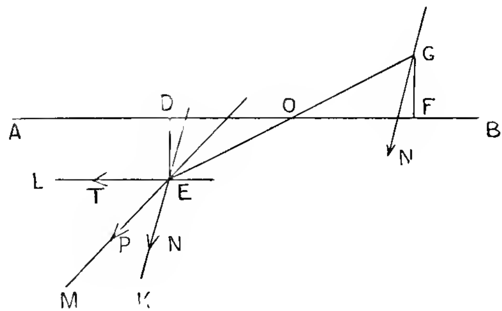


Fig. 62.

D E about the axis A B, and it has been shown that when D is a fixed point, N may be replaced by any other force in the plane of N and D E which has the same moment about D. Again, let Q be a force whose moment about the fixed axis is equal and opposite to that of P ; then Q may first be resolved into two components, one of which is parallel to A B and the other in a perpendicular direction ; the latter only of these being retained, since the former is without effect. This latter component, perpendicular to A B, may further be replaced by a force of equal moment acting at the end of an arm F G, which arm is equal and oppositely directed to D E ; and since the moment of Q about A B is equal and opposite to that of P, the force acting at G will be equal and *similarly directed* to N. Now

it is evident, since D E and F G are equal and parallel, that A B, D E, and F G all lie in one plane, and that D F, E G bisect one another at O. Through this point, then, passes the resultant of the equal and similarly-directed forces N, N ; their combined action having, therefore, no tendency to rotate the body about A B. But N, acting at E, was shown to be equivalent in effect to P, and N, acting at G, was also proved equivalent to Q ; hence, P and Q, acting together on the constrained body, will be in equilibrium.

We adopt, then, the following definition : *The moment of a force about an axis is the product of the common perpendicular between the axis and the line of action of the force, multiplied by the resolved part of the force in a direction at right angles both to this perpendicular and to the axis.*

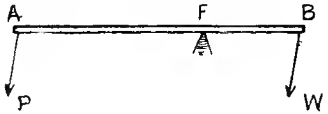


FIG. 63, *a*.

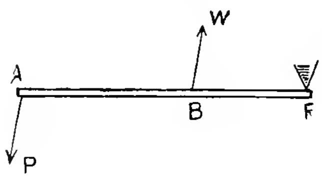


FIG. 63, *b*.

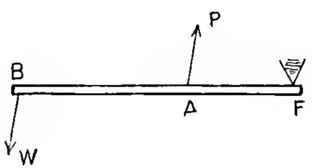


FIG. 63, *c*.

It is also evident that *The resultant moment of any number of forces acting on a rigid body so constrained, is equal to the algebraic sum of the moments due to separate forces* ; and that, if the algebraic sum of the moments is zero, the forces will be in equilibrium

159. **The Lever** *is a rigid rod capable of turning about a fixed point, which is called the* **fulcrum.**

Generally speaking, the forces applied to the lever lie in one plane, so that one condition is sufficient for equilibrium : *The algebraic sum of the moments about the fulcrum must vanish.*

In the case of two parallel forces applied at the extremities of a straight lever A B, *the forces* (P, W) *must be inversely proportional to the arms* (A F, F B) *on which they act* (fig. 63, *a*, *b*, *c*).

This follows from equation (19) of §122 (for, when the resultant passes through the fulcrum, we know that there will be equilibrium). It is sometimes referred to as 'the Principle of the Lever.'

Levers are divided into three classes, according to the relative positions of the fulcrum (F), the power (P), and the weight (W).

When the fulcrum is between the power and the weight (fig. 63, *a*), *the lever is one of the* **first class**, and the weight may obviously be either greater or less than the power.

In levers of the **second class**, *the weight acts between the power and the fulcrum* (fig. 63, *b*), and is always greater than the power.

In levers of the **third class**, *the power lies between the fulcrum and the weight* (fig. 63, *c*), and is necessarily greater than the weight.

The mechanical advantage is evidently A F/F B, the ratio of the arms of the lever.

160. **Combination of Levers.**—It is sometimes found useful in practice to obtain increased advantage by the combined action of two or more levers. Thus, let A B, B C, C D be

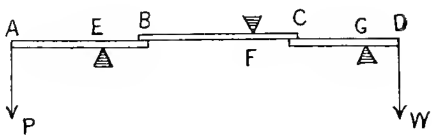


Fig. 64.

three levers, whose fulcrums are at E, F, and G respectively (fig. 64). A downward force P at A will produce an upward force P · (A E/E B) at B on the lever B C; this, again, will produce a downward force P · (A E/E B) · (B F/F C) at C on the lever C D; and this, in turn, will give rise to an upward force P · (A E/E B) · (B F/F C) · (C G/G D) at D, which, for equilibrium, must be balanced by a downward force W of equal magnitude; *the mechanical advantage*, W/P, *being equal to the continued product of the mechanical advantages of* A B, B C *and* C D.

161. **The Wheel and Axle.**—If an axle, A B, has two arms, C D, E F, of lengths m and n, each perpendicular to it (fig. 65), and if a force P act at D perpendicularly to C D and to A B, and a force W at F perpendicularly to E F and to A B, equilibrium will obtain when (but not unless) P and W act in opposite directions of rotation about A B, while P m is equal in magnitude to W n; or, in other words, each moment being taken with its proper sign, we must have

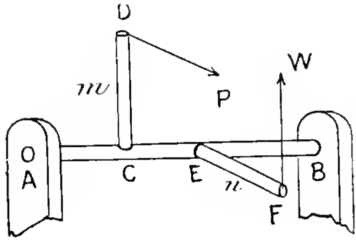


FIG. 65.

$$P\,m + W\,n = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

This is the principle of the wheel and axle, the construction of which is as follows (fig. 66): A cylindrical drum, D, of radius R, and a spindle, S, of smaller radius, r, are rigidly connected, so that their geometrical axes lie in one straight line about which they can freely rotate together. A rope or cord is attached at one end to the drum D, and is wound several times around it, the free end being pulled downwards by a force P. A cord is similarly attached to S, but is so wound that a rotation of the cylinders which winds up one cord unwinds the other, and its free extremity is pulled downwards by a force W. Now, each of the forces P and W tends to unwind the string on which it acts, and the moments are therefore in contrary directions about the fixed axis A B. The arm on which either force acts s a radius of the cylinder on which the corresponding cord is wound; and hence, for equi-

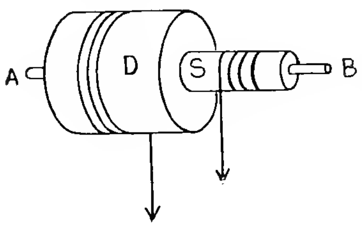


FIG. 66.

librium, PR must be equal in magnitude to Wr. Thus the mechanical advantage

$$W/P = R/r \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

162. **The Windlass** (fig. 67).—Here the 'power' P, instead of acting on a string wound round a cylindrical drum, is applied to a handle H at the end of an arm of length l; the 'weight' W acts as in the wheel and axle, and, evidently, the mechanical advantage

$$W/P = l/r \ldots(30)$$

where r is the radius of the cylindrical drum.

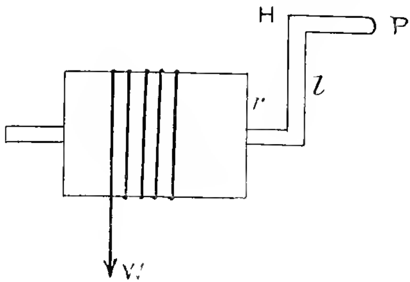


FIG. 67.

163. **The Differential Axle** (fig. 68).—In this machine a cord passes round a pulley, A, so as to support it, while the two ends of the cord are wound round two cylinders, C and D, which are rigidly connected together so as to have a common geometrical axis; and the winding is such that, when the handle is turned, the string is wound up on one cylinder and unwound from the other. A handle, H, at the end of an arm

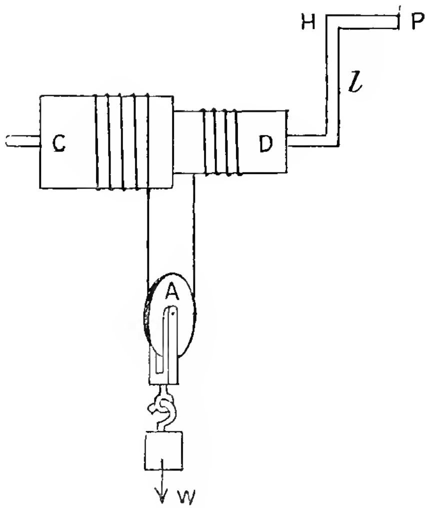


FIG. 68.

of length l, being acted on by a force P, prevents the descent of the pulley, which sustains the weight W. Let c and d be the radii of the respective cylinders; then the pull exerted by the string on each side of the pulley is equal to $\frac{1}{2}$W, and the moment around the axis of the cylinders is $\frac{1}{2}Wc - \frac{1}{2}Wd$. For equilibrium we must have Pl equal in magnitude to $\frac{1}{2}W(c-d)$; whence the mechanical advantage

$$W/P = 2l/(c-d) \qquad \textbf{(31)}$$

When the handle is turned, a certain length of the cord will be wound up on C, and a smaller length will be wound off of D; the hanging portion of the cord will then be shortened by the difference of these lengths, and the pulley and suspended weight will evidently be raised by half this amount.

164. **The Toothed Wheel** (fig. 69).—Let A and B be two toothed wheels of radii r_1 and r_2, each of which can turn about a fixed axis perpendicular to its plane; and suppose, further,

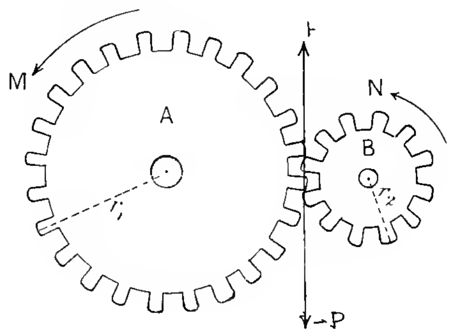


Fig. 69.

that the length of arc between successive teeth is the same for each wheel. Let forces be applied to A having a moment about its axis equal to M, and let forces be applied to B having a moment about its axis equal to N; then, if a certain relation hold between M and N, the system will be in equilibrium.

In the first place, it is evident that M and N must tend in the *same* direction of rotation, as indicated in the figure, and

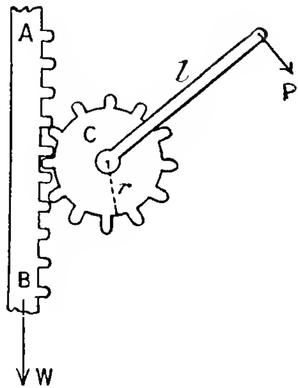
the force which the wheels exert upon one another may be considered to act along their common tangent; for, if there were any component in a direction perpendicular to this tangent, it could not affect the equilibrium. If P is the force exerted by A upon B, $-$P will be the force exerted by B upon A. That each wheel may be in equilibrium, the algebraic sum of the moments about its fixed axis must vanish; that is,

$$Pr_1 = -M, \quad Pr_2 = -N;$$

or,

$$M/N = r_1/r_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

By successive applications of this principle it is easy to deduce the mechanical advantage of a train of wheelwork, as measured by the ratio of the equilibrating moments applied to the first and last wheels.



IG. 70.

165. **The Rack and Pinion** (fig. 70).—The rack, A B, is a straight toothed rod, which can only move backwards and forwards along the direction of its length; while the toothed wheel C can only turn about an axis through its centre. The 'weight' W acts down wards on the rack A B, while the 'power' P, acting at the end of an arm of length l, has a moment Pl about the centre of C, whose radius may be called r. For equilibrium, the sum of the moments about C must vanish; so that Pl must be equal in magnitude to Wr, and the mechanical advantage

$$W/P = l/r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

166. **The Screw.**—If a smooth cylindrical rod, A B, is enclosed for a part of its length in a fixed case, C, which fits it exactly but without friction (fig. 71), the rod will be capable of two kinds of motion; it can slide either way parallel to its axis, or it can rotate about that axis. If the cylindrical rod

were furnished with projections running parallel to its axis (fig. 72), which fitted into corresponding grooves or depressions in the concave surface of C, all movement of rotation would be prevented, and the rod A B could only slide backwards and forwards. If, on the other hand, its surface were encircled by a number of projecting rings (fig. 73) fitting into corresponding grooves in the surface of C, A B would be free to rotate, but could not move along in the direction of its axis. Finally, suppose that there is a single uniform projection or 'thread' running spirally around the cylindrical rod (fig. 74), and fitting into a spiral groove in the inner surface of C. Then, C being fixed, A B can move along in the direction of its axis at the same time that it rotates, but its motion is constrained in just the same degree as in the two previous cases ; for the distance through which it advances is exactly determined by the angle through which it is turned, and *vice versâ*. A B is now a **screw**, the case C in which it turns being called the **nut**: *A screw is a body which, by its form, is so constrained that it can only move parallel to a certain fixed axis while at the same time it rotates about that axis ; the longitudinal displacement being proportional to the rotation.* This, of course, supposes the nut to be fixed ; when the screw is fixed, the displacement of the nut follows exactly the same law.

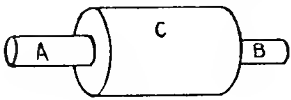


FIG. 71.

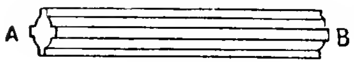


FIG. 72.

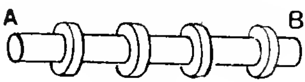


FIG. 73.

FIG. 74.

167. **Pitch.**—It is evident that a rotation through four right angles (that is, one complete turn) will cause the screw to advance through the distance between two consecutive turns of the thread ; this distance is called the *pitch* of the screw.

Now, let A B be a cylinder (fig. 75) and K L M a triangle, right-angled at M, whose base, K M, is equal to the circumference of the cylinder. If the side L M of this triangle were fixed to the surface of the cylinder at L′ M′, so as to lie parallel to the axis, and if the triangle were then wrapped round the cylinder, its base would coincide with a circumference, and the angular point K would be at K′, coincident with M′. The hypotenuse K L would also be wrapped round the cylindrical surface, and would intersect the generating lines[1] at an angle equal to K L M; so that if the axis of A B were vertical, the helical line K′ H′ G′ L′ would be inclined to the horizontal at a constant angle equal to M K L, and would evidently be the form of one turn of a screw-thread of pitch L M on the cylinder A B.

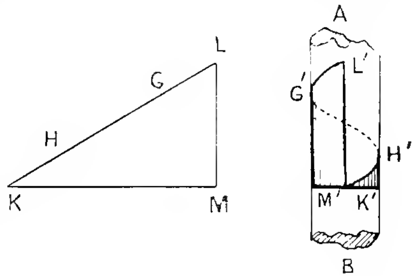


Fig. 75.

168. **Equilibrium of the Screw.**—Let A B (fig. 76) be a portion of this screw (which we shall suppose perfectly smooth), and let the nut be fixed in such a position that the axis of the screw is vertical. If, now, a downward force W act on the screw, the thread will experience a reaction from the groove in which it fits, and this reaction will be at each point perpen-

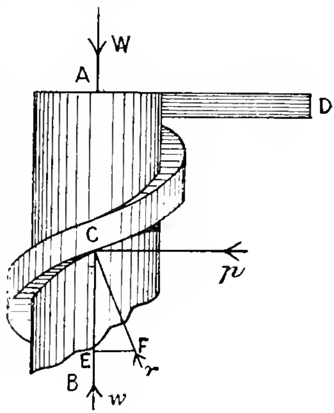


Fig. 76.

[1] Lines drawn on the surface of a cylinder parallel to its axis are called *generating lines*.

dicular to the surface of the thread. Thus at each point, such as C, the force exerted on the thread may in general be resolved into two components at right angles to one another, and both perpendicular to the length of the thread. Let one of these components be taken along the perpendicular at C to the surface of the cylinder, so that the other component r is tangential to that surface, as indicated in the figure. Since the screw cannot move in a direction perpendicular to its axis, the components of force perpendicular to the cylindrical surface must, on the whole, balance one another, and may be left out of account. Each such component as r, on the other hand, may be further resolved into two components: one, w, along a generating line of the cylinder; the other, p, perpendicular to w and tangential to the cylindrical surface, that is, perpendicular to the radius drawn to C.

The tendency of the force p is twofold: (1) to displace the screw in a direction perpendicular to its axis, a tendency which may be left out of account, since such displacements cannot occur; (2) to rotate the screw about its axis, the moment of p being $p\,a$, where a is the radius of the cylinder.

Now let $\Sigma\,(w)$ denote 'the sum of all such forces as w,' and $\Sigma\,(p\,a)$ 'the sum of all such moments as $p\,a$'; thus, owing to the reactions of the nut, the screw experiences an upward force equal to $\Sigma\,(w)$, and a moment about its axis equal to $\Sigma\,(p\,a)$, that is, equal to $a \cdot \Sigma\,(p)$; for a, the radius of the cylinder, is the same in every case.

To find the relation between p and w, complete the triangle C E F by drawing E F horizontal; then $r : p : w =$ F C : F E : E C. But F C E is the complement of the angle at which the thread cuts the generating lines of the cylinder, and is, therefore, equal to the angle L K M (fig. 75), and the two right-angled triangles F C E, L K M are similar to one another, and L M/M K $=$ F E/E C $= p/w$. At each point of the thread there will be the same ratio between the forces corresponding to p and to w, and we shall therefore have: $\Sigma(p)/\Sigma(w)$ $=$ L M/M K $=$ pitch/circumference.

The downward force W acting on the screw must be equal and opposite to $\Sigma\,(w)$, and to balance $\Sigma\,(p)$ we shall require a

horizontal force whose moment about the axis is equal and opposite to $a \cdot \Sigma(p)$. Let this horizontal force be P, acting perpendicularly at the end of an arm A D, whose length (measured from the axis) is l; then,

$$P\,l = -\,a \cdot \Sigma(p) \text{ and } W = -\,\Sigma(w)\,;$$

$$\therefore P\,l/W = a \cdot \Sigma(p)/\Sigma(w) = a \times \text{pitch}/\text{circumference}\,;$$

$$\therefore \frac{W}{P} = \frac{l}{a} \cdot \frac{2\pi a}{\text{pitch}} = \frac{2\pi l}{\text{pitch}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(34)}$$

or, *the mechanical advantage is equal to the circumference described by* P *divided by the pitch of the screw.*

If P is applied tangentially to the surface of the cylinder, the arm on which it acts is the radius a, the circumference which it describes being equal to K M, and the mechanical advantage is then K M/M L; *which is the same as if a heavy body were kept in equilibrium by a horizontal force against a smooth inclined plane*, K L.

In an exactly similar manner it might be shown that if the force sustaining the screw were applied along the direction of the thread, *the mechanical advantage would be the same as if a heavy body were supported on the smooth inclined plane*, K L, *by a force acting along the plane.*

169. **Right-handed and Left-handed Screws.**—All the screws represented in the foregoing figures are right-handed. *A body has a right-handed screw-motion when it recedes from the observer while rotating in the direction of clock-hands*, and it should be remarked that the relation thus defined is essential to the motion and does not depend on the position of the observer. For, if the same object were viewed from the opposite direction, it would *approach* the observer while rotating in the *contrary* direction to clock-hands.

If a fixed nut constrains a screw to execute a right-handed motion, then the motion of the nut must be right-handed when the screw is fixed. For, suppose we are looking at the screw 'end on,' and that it is rotating in the direction of clock-hands; it will be receding from us. Now let the screw be fixed and the nut turned in the *contrary* direction to clock-hands; the *relative* motion of the screw and the nut is thus unchanged in

direction, so that the nut is now approaching us with a counter-clockwise rotation. Its screw-motion is therefore right-handed.

Examples on Chapter XI.

(1) A smooth plane is inclined to the vertical at 60°, and a body whose weight is W rests against the plane. Find (*a*) the horizontal force; (*b*) the force parallel to the plane which must act upon the body to preserve equilibrium.

(2) In the previous question, suppose that equilibrium is preserved by two equal forces, one horizontal, the other parallel to the plane, and find the magnitude of either force.

(3) If a horizontal force of 400 g dynes can just support a mass of 700 grams against a smooth inclined plane, find the ratio of the base to the height of the plane.

(4) Two smooth planes are placed back to back so that they have a common height, while their lengths are to one another as 3 : 7. A string passes over a smooth pulley at the vertex of the planes, and its ends are attached to two heavy bodies, which rest one on each of the smooth planes. What must be the ratio of the weights of the two bodies that equilibrium may obtain?

(5) A body of weight W is attached to a single movable pulley of weight w, and each part of the string passing round the pulley is inclined at 60° to the vertical. Find the pull of the string.

(6) There are three movable pulleys whose masses are 50, 60, and 80 grams respectively, and they are arranged according to the first system, the 'weight' acting downward on the 80-gram pulley, and the 'power' acting upward on a string passing round the 50-gram pulley. If the power is a force of 400 g dynes, what is the greatest ratio which the 'weight' can bear to the sum of the weights of the pulleys?

(7) A rope passes over a smooth fixed pulley, and a man sits in a cage which is fastened to one end of the rope, while he holds the other end in his hand. What is the least force which the man must exert in order to raise himself from the ground?

(8) Suppose that a man were fastened by ropes to the pulley B (fig. 61), with what force must he pull the free end of the string in order to raise himself?

(9) If a lever of the first class is 3 metres in length, where must the fulcrum be placed that the mechanical advantage may be 9?

(10) Two arms of a rigid lever make an angle of 120° with one another, and one arm has twice the length of the other. Neglecting the weight of the lever, find the ratio of the masses which must be attached to its ends that it may rest with its shorter arm horizontal.

(11) A lever is pivoted at one end, and a string attached to its middle point is pulled with a force P in a direction making an angle of 30° with the lever. Find the smallest force which can be applied to the other end of the lever so as to preserve equilibrium.

(12) The two arms of a lever are equal, and at right angles to one another. Find the position of equilibrium of the lever when masses of 10 and $10\sqrt{3}$ lbs. are hung from its respective ends.

(13) Being given two or more forces which maintain a lever in equilibrium, show how to find the position of the fulcrum.

(14) One end of a straight lever is pivoted against a vertical wall, and a string is also attached to the wall at a point 1 metre above the pivot; the other end of this string is attached to the further end of the lever, which is thus maintained in a horizontal position. If the weight of the lever acts at its middle point and is equal to the pull of the string, find the length of the string and of the lever.

(15) In a wheel and axle the 'power' descends a foot for every inch through which the 'weight' is raised. What is the mechanical advantage of the machine?

(16) The two ends of a rope are wound round a differential axle, while the intervening loop passes over a fixed pulley from which the axle is thus suspended. If W is the weight of the axle, *a* and *b* the radii of its two portions, and *r* the radius of the pulley, find what moment must be applied to the pulley in order to preserve equilibrium, the axle resting horizontally.

(17) A handle half a metre in length is fixed on the same axle with a toothed wheel 10 centimetres in diameter; this in turn is geared into another wheel 40 centimetres in diameter, turning on the same axle with a windlass-drum 25 centimetres in diameter, round which one end of a rope is coiled. If the 'weight' is suspended by this rope while the 'power' is applied to the handle, find the mechanical advantage of the machine.

(18) If a moment of 1,000,000 dyne-cm. around the axis of a screw can balance a force of 10,000,000 dynes acting along the axis, what is the distance between two consecutive turns of the thread?

(19) A screw-thread on the surface of a cylinder cuts the generating lines at an angle of 60°. What will be the mechanical advantage when the power acts tangentially to the cylinder and at right angles to the generating lines?

(20) The axis of a screw is vertical, and the point of application of the 'power' describes a path whose inclination to the vertical is θ. If the power acts horizontally, what is the mechanical advantage?

CHAPTER XII

MOTION ON AN INCLINED PLANE—FRICTION

170. **Motion on a Smooth Inclined Plane** (fig. 77).—It has already been seen (§ 143), that a body of weight W is in equilibrium on a smooth inclined plane when a force $P = (h/l)$ W (or W sin α) acts upon it upwards along the plane, the normal resistance of the plane being at the same time R $= (b/l)$ W (or W cos α). We can therefore resolve W into two components, one equal and opposite to R, the other equal and opposite to P. If the force P be removed, there will no longer be equilibrium, for though R will still balance that component of W which is perpendicular to the plane, there will be left an unbalanced component of magnitude P acting down the plane, and this will produce acceleration in the body.

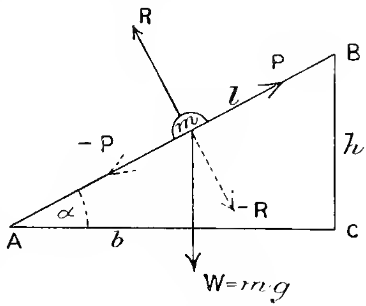


FIG. 77.

The mass being m, the weight W will be equal to mg and $P = W\ (h/l) = mg\ (h/l)$, or mg sin α. Hence the acceleration, which is directed down the plane,

$$f = (mg \sin \alpha)/m = g \sin \alpha \ldots\ldots \qquad \ldots\ldots(\mathbf{35})$$

that is, h/l or sin α is the ratio of the acceleration on the inclined plane to the acceleration which a body would have when falling freely under the action of its weight. When the

plane is horizontal, h/l or $\sin \alpha$ is zero, and the body is without acceleration; in other words, it will either remain at rest, or, if moving, will continue to move in a straight line with uniform velocity. As α increases, so does the acceleration $g \sin \alpha$, until, finally, when the plane is vertical, α becomes a right angle and h/l, or $\sin \alpha = 1$; f is now equal to g, the plane having no influence on the motion of the body.

171. The treatment of Chapter VII., on Projectiles, is evidently applicable to the motion of bodies on a smooth inclined plane, for in each case the motion takes place in one plane, and in each case the acceleration is constant in direction and magnitude. In general, the path of a body moving on a smooth inclined plane will be a parabola whose directrix is horizontal and whose vertex is at its highest point; the axis being parallel to the direction A B (fig. 77) determined by the intersection of the given plane with a vertical plane perpendicular to it.

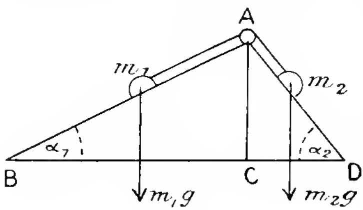


FIG. 78.

172. Two smooth fixed planes A B, A D (fig. 78) are inclined to the horizontal at angles A B C ($= \alpha_1$) and A D C ($= \alpha_2$) respectively. A mass m_1 resting on A B, and a mass m_2 on A D, are connected by a fine, flexible, inextensible string passing over a smooth fixed pulley at the common vertex A of the planes. The pull T of the string acts on each body in an upward direction along the corresponding plane, and we have for the respective accelerations *down* the planes

$$\frac{1}{m_1}(m_1 g \sin \alpha_1 - T) \text{ and } \frac{1}{m_2}(m_2 g \sin \alpha_2 - T).$$

Since the string is inextensible, these downward accelerations are equal and opposite; that is,

$$m_2(m_1 g \sin \alpha_1 - T) + m_1(m_2 g \sin \alpha_2 - T) = 0;$$

thus, $(m_1 + m_2)\,T = m_1\,m_2\,(g \sin \alpha_1 + g \sin \alpha_2)$;

or, $$T = \frac{m_1\,m_2}{m_1 + m_2}\,g\,(\sin \alpha_1 + \sin \alpha_2) \;\ldots\ldots\ldots\ldots(36)$$

The acceleration of m_1 down the plane A B is

$$f_1 = \frac{1}{m_1}\,(m_1\,g \sin \alpha_1 - T)$$

$$= \frac{1}{m_1}\left(m_1\,g \sin \alpha_1 - \frac{m_1\,m_2}{m_1 + m_2}\,g\,(\sin \alpha_1 + \sin \alpha_2)\right);$$

i.e. $$f_1 = \frac{m_1 \sin \alpha_1 - m_2 \sin \alpha_2}{m_1 + m_2}\,.g \;\ldots\ldots\ldots\ldots(37)$$

and this acceleration will be positive (that is, *down* the plane A B) provided $m_1 \sin \alpha_1$ is greater than $m_2 \sin \alpha_2$.

Some particular cases of this problem may be noticed :

(1) If α_1 and α_2 are both right angles, the planes are vertical and the system is dynamically equivalent to an Atwood's machine. It will accordingly be seen when $\sin \alpha_1$ and $\sin \alpha_2$ are each made equal to unity, that equations (36) and (37) become identical with (15) and (16) of § 102.

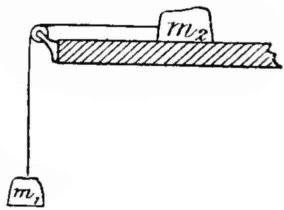


FIG. 79.

(2) If $\alpha_1 = 90°$ while $\alpha_2 = 0$, the body m_1 will be hanging vertically, and will draw m_2 along a smooth horizontal table (fig. 79). In this case $\sin \alpha_1 = 1$ and $\sin \alpha_2 = 0$, and we find for the acceleration

$$f_1 = \frac{m_1 g}{m_1 + m_2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

and for the pull of tne string

$$T = \frac{m_1\,m_2}{m_1 + m_2}\,g \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39)$$

It may be remarked that the latter quantity is half as great as it would be if both bodies were hanging vertically, and is unaltered by interchanging m_1 and m_2.

173. **Time of Descent.**—It is evident that the laws of motion along a smooth inclined plane are also applicable to a

particle moving in a smooth straight tube, or a bead sliding on a smooth straight wire, and in general to all cases where a body is compelled by smooth, rigid constraints to follow a rectilinear path inclined to the horizontal at any angle (θ). In each case the acceleration down the path is $g \sin \theta$, and the distance s will be described from rest in the time

$$t = \sqrt{\frac{2s}{g \sin \theta}} \quad \text{......................(40)}$$

When we speak of the time of descent along a given path, we understand that the body starts from rest, and is constrained to describe that path under the action of gravity. Thus it may be shown that, if a number of chords be drawn from the highest point of a vertical circle, the time of descent down each is the same.

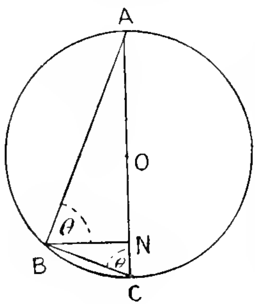


Fig. 80.

For, let A B C (fig. 80) be the vertical circle, and from A, its highest point, draw any chord A B and also the diameter A O C, which latter is obviously vertical. Join B C, and draw B N horizontal to meet A C at N.

Let the angle N B A $= \theta$; then, since A B C, the angle in a semicircle, is a right angle, C B N is the complement of θ and N C B $= \theta$. Thus, B A/C A $= \sin \theta$, or A B $= d \cdot \sin \theta$, where d is the diameter of the circle; and the time of descent down the cord A B

$$t = \sqrt{\frac{2d \sin \theta}{g \sin \theta}} = \sqrt{\frac{2d}{g}},$$

which the same as the time of descent down the vertical diameter A C.

This will also be the time of descent along all chords through the *lowest* point of the circle, since for every such cord there is an equal and equally inclined cord through the highest point.

174. *To find the straight line of quickest descent from a given point to a given straight line in the same vertical plane.*—

Let A be the given point, M N the given straight line (fig. 81). Describe a circle having its highest point at A, and touching M N at B; then A B is the required straight line of quickest descent.

For, let A D be any other straight line drawn to meet M N, and cutting the circumference of the circle in E. The time of descent along A B is equal to that along A E by § 173, and hence the time along A B is less than that along A D. Similarly, it may be shown that the time of descent along A B is less than that along any other straight line drawn from the point A to the straight line M N.

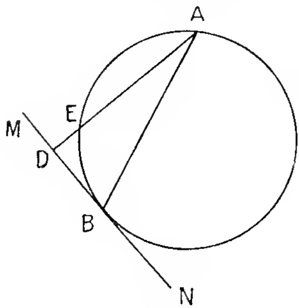


FIG. 81.

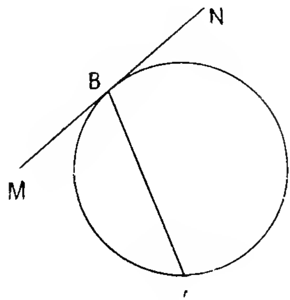


FIG. 82.

175. *To find the straight line of quickest descent from a given straight line to a given point lying in the same vertical plane.*—Let M N be the given straight line, A the given point (fig. 82). Describe a circle having its lowest point at A and touching M N at B. Then, by reasoning similar to that of § 174, it is easily shown that B A is the straight line of quickest descent from M N to the point A.

176. **Friction.**—The condition of perfect smoothness defined in § 142 and frequently assumed in later sections, can never be realised in practice. When the surfaces of two solid bodies are moved over one another, the force exerted by the one upon the other will always have a tangential component T, as well as a normal component N (fig. 83), the direction of T

being always such as to oppose the sliding motion. It was stated in § 142 that T could not exceed a certain limiting value, and that if B were pulled by a force greater that this, sliding would take place. The limiting value of T depends on the nature of the surfaces in contact, and also on the value of N ; the rougher the surfaces, and the more firmly they are pressed together, the greater is the force required to overcome friction and produce sliding. Experiment shows the following laws to be approximately correct :

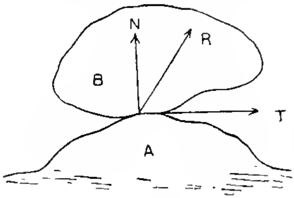


FIG. 83.

(1) *Friction acts in the direction opposed to that in which sliding takes place, or tends to take place.*

(2) *Friction is proportional to the normal force between the sliding surfaces.*

(3) *Friction is independent of the extent of the surfaces in contact, when the normal force is given.*

(4) *Friction is nearly independent of the velocity of sliding.*

(5) *Friction is proportional to a co-efficient* μ, *depending on the nature of the sliding surfaces.*

177. **Coefficient of Friction.**—Let a body of mass m rest on a rough horizontal plane (fig. 84) ; the normal reaction of

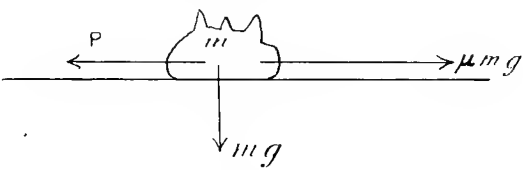


FIG. 84.

the plane which supports the body is equal and opposite the weight $m g$, and if $\mu m g$ is the least horizontal force which will produce sliding, μ is called the *coefficient of friction* for the surfaces in contact. If any smaller horizontal force is applied to the body, it will be balanced by an equal and opposite force

due to friction; but if the impressed horizontal force has any value greater than $\mu m g$, sliding will take place, the friction being opposed to the motion and equal to $\mu m g$, and, so long as there is any sliding motion, the friction will retain this magnitude unaltered.

If the body m were to rest on the plane in any other position, so as to make the surface of contact larger or smaller, the coefficient of friction μ, and the limiting value $\mu m g$ of the frictional resistance, would still be approximately the same. In extreme cases, however, this would no longer be true; for, suppose the body is made to rest on three hard, sharp spikes, and that the plane surface belongs to a soft substance. The friction would now be much greater than before, the coefficient μ would have ceased to have any definite value, and the body would have a more or less jerky and spasmodic motion when drawn by a constant horizontal force. That the laws of friction may approximately hold, the surfaces in contact must not yield to an appreciable extent; but the truth of these laws will always be subsequently assumed; that is, we shall suppose the necessary physical conditions to be fulfilled.

178. **Equilibrium on a Rough Inclined Plane** (A B, fig. 85).—Draw A C vertical and B C horizontal, and, as usual, let B A $= l$, B C $= b$, C A $= h$. Let a mass m be in equilibrium on the plane under the action of its weight W ($= m g$), the resistance R perpendicular to the plane, and a force P acting along the plane. Then from § 143 we know that $P/R = h/b$, or $P = R \cdot h/b = R \tan \alpha$, where α is the

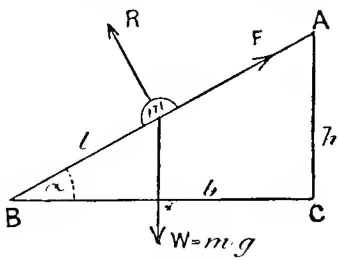


Fig. 85.

inclination of the plane to the horizon. Now, if h/b or $\tan \alpha$ is *not greater than* μ, the coefficient of friction, the body when left to itself will remain in equilibrium on the plane; but if h/b is *greater than* μ, there will not be enough friction to prevent the body sliding down the plane, and an additional force

at least as great as $[(h/b) - \mu]$ R must act up the plane to preserve equilibrium. If a greater force than this is applied equilibrium will still obtain, *unless the force is greater than* $[(h/b) + \mu]$ R, in which case the frictional resistance (acting down the plane) would require for equilibrium to be greater than μ R.

179. **Angle of Friction.** – If the plane A B were at first horizontal, and were then gradually tilted, h/b would be at first equal to zero, and would then continually increase, and when h/b exceeded the value μ, the body would begin to slide down the plane. The critical inclination (θ) of the plane when sliding is just about to commence, is called the *angle of friction*, and we have consequently :

$$\mu = \tan \theta \qquad (41)$$

As an illustration, let a coin be placed on a book, which is then very slowly tilted until the coin slides off. If μ were quite constant, the acceleration of the coin should be inappreciably small when the inclination only just exceeded the angle of friction ; it may be found, however, that once sliding has commenced, the acceleration is very appreciable, and this is due to the fact that the coefficient of friction between the surfaces of solids is usually greater at rest than when sliding actually takes place. *Just before* the coin slides the friction is opposed to the greatest force consistent with equilibrium ; as soon as sliding commences, the coefficient of friction diminishes, and there is left a finite component of force urging the body down the plane.

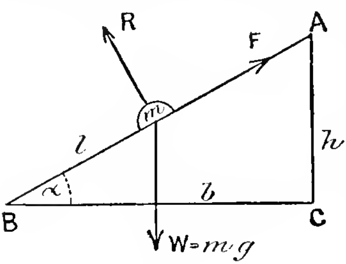


Fig. 86.

180. **Motion on a Rough Inclined Plane,** the coefficient of friction being constant and equal to μ (fig. 86).

(a) Let the body be moving *down* the plane ; then the forces acting upon it are its weight W ($= mg$), the resistance R perpendicular to the plane, and the friction μ R, acting up

the plane. The total component of force acting *down* the plane is thus

$$mg \sin \alpha - \mu R = mg \sin \alpha - \mu \cdot mg \cos \alpha\,;$$

the first term being the resolved part along the plane of the weight mg. The acceleration of the mass m is therefore

$$f = \frac{mg\,(\sin \alpha - \mu \cos \alpha)}{m} = g\,(\sin \alpha - \mu \cos \alpha) \ldots\ldots (42)$$

If α is *greater* than the angle of friction, $\sin \alpha$ will be greater than $\mu \cos \alpha$, and the downward acceleration will be *positive*; if α is *less* than the angle of friction, $\sin \alpha$ will be less than $\mu \cos \alpha$, and the downward acceleration will be negative; so that the motion of the body is retarded, and it will finally come to rest.

(*b*) Let the body be moving *up* the plane; the component of force *down* the plane is now $mg \sin \alpha + \mu \cdot mg \cos \alpha$, for the friction is still opposed to the motion, while the other forces have the same direction and magnitude as before; thus the motion will be retarded, the downward acceleration being

$$f = g\,(\sin \alpha + \mu \cos \alpha) \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

After the body has come to rest, it will move down the plane with the acceleration $g\,(\sin \alpha - \mu \cos \alpha)$ if α is greater than the angle of friction, or, in the contrary case, will remain at rest.

Examples on Chapter XII.

(1) Show that the velocity acquired in descending from a point A to a point B on a smooth inclined plane depends only on the vertical height of A above B.

(2) If a smooth inclined plane be of given height, show that the time of descent down the plane is proportional to its length.

(3) A rough plane is inclined 60° to the horizon, and it is found that when a certain body is ascending the plane the downward acceleration is twice as great as when descending · find the coefficient of friction.

(4) Two bodies are connected by a string which passes over

a smooth pulley at the edge of a rough horizontal table; one of the bodies (mass m) rests on the table, while the other (mass m') hangs vertically; and it is found that, when this latter body descends, the motion of the system is uniform. Determine the coefficient of friction in terms of m and m'.

(5) If the planes A B, A D in fig. 78 are rough, and have coefficients of friction μ_1, μ_2 respectively, obtain an expression for the acceleration when the system moves in either direction, and also for the pull of the string.

(6) A body lies on a rough horizontal plane, and is acted on by a force just sufficient to move it, the direction of the force being inclined upward at 45° to the horizon. If μ is the coefficient of friction and W the weight of the body, find the value of the force.

(7) In fig. 79, find the velocity of the body m_1 relatively to m_2.

(8) In fig. 79, find the velocity of the point half-way between m_1 and m_2.

(9) A body is projected with velocity v up a rough plane whose inclination to the horizon is equal to the angle of friction. To what vertical height will it ascend?

(10) Find the coefficient of friction between a body and a rough plane, if, when the plane is inclined 45°, the body ascends with a retardation of magnitude g.

CHAPTER XIII

CENTRE OF GRAVITY—BALANCES

181. **Centre of Parallel Forces.**—Let there be any number n of parallel forces, P_1, P_2, . . . P_n, acting upon a rigid body at the points A_1, A_2, . . . A_n respectively (fig. 87). First take two of the forces P_1 and P_2; their resultant is a parallel force $P_1 + P_2$ acting at a point B_2 on $A_1 A_2$, such that $A_1 B_2/B_2 A_2 = P_2/P_1$. The resultant of $P_1 + P_2$ at B_2 and P_3 at A_3 is again a parallel force, $P_1 + P_2 + P_3$ acting at a point B_3 on $B_2 A_3$, such

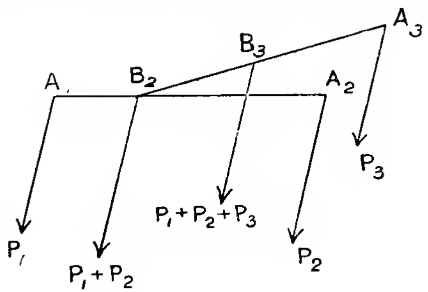


Fig. 87.

that $B_2 B_3/B_3 A_3 = P_3/(P_1 + P_2)$; P_3 not being necessarily in the same plane as P_1 and P_2. Thus we may proceed till all the forces have been compounded into a single resultant $P_1 + P_2 + \ldots + P_n$ acting at a point B_n, which is called *the centre of the parallel forces.* As a matter of fact, though the *line of action* of the resultant is determinate when the forces are given, we may suppose the resultant to be applied at any point along this line; but the centre of parallel forces as just defined has certain important properties.

First, we must show that the centre depends only on the positions of the points A_1, A_2 . . . and on the signs and magnitudes of the corresponding parallel forces, being independent alike of the direction to which all the forces are parallel, and of the order in which they are compounded. For it is evident that the plane through A_1, A_2 and A_3 contains the points B_2 and B_3; and hence B_3 is the point where this plane is intersected by the resultant of P_1, P_2 and P_3. Now, the resultant of these three forces being exactly equivalent to them in effect must necessarily be the same, in whatever order they may be compounded; and since the plane through A_1, A_2, A_3 is determined by the positions of these three points, the position found for B_3 will be the same in whatever order P_1, P_2, P_3 are compounded; by successively making changes of this kind in the order of composition of the forces, the final order may be any we please, and the position of the point B_n will remain the same as at first. Further, it is evident that the position of B_2 depends only on those of A_1 and A_2 and on the ratio of the *magnitudes* of P_1 and P_2, *not* on their common direction; and, proceeding in this way, it will be seen that the position of B_n is independent of the common direction of the n parallel forces.

Hence, *When a number of like parallel forces are given in magnitude, and act at given points, their centre is completely determinate.*

182. **Centre of Gravity.**—A material body may be conceived of as made up of a large number of very small particles, each of which has a certain small mass of its own, and a weight equal to g times this mass, where g is the local value of the acceleration due to gravity. The weights of all these particles constitute a system of parallel forces such as we have been considering, and the resultant, which is equal to their sum, is called the weight of the body. The centre of the system of parallel forces is called the *centre of gravity* of the body, and from § 181 it follows that the centre of gravity is a point fixed in the body, and does not vary when the body is turned into another position; for although the forces on the constituent particles are then differently inclined to the figure of the body, they remain parallel to one another, and their magnitudes and

points of application are also unchanged. We adopt, then, the following definitions : *The weight of a body is the resultant of the weights of the separate particles of which we may conceive the body to be composed, and the centre of gravity of a body is that point through which its weight always acts.*

If the centre of gravity be fixed, so that the body is free to turn in any direction under the action of its own weight, it will exhibit no tendency to do so ; for the weight is a force acting *through* the centre of gravity, and its moment about this point is therefore zero. The centre of gravity, then, may also be defined as *that point about which the body will balance in any position* ; and, at the same time, we may enunciate the following theorem :

So far as the action of gravity is concerned, a body may be replaced by a single particle of the same mass, situated at its centre of gravity ; or, in other words, we may suppose the whole mass of the body to be collected at its centre of gravity ; for this merely amounts to replacing a number of component forces (weights of the particles) by their resultant (weight of the body).

183. **Centre of Gravity of a Straight Line.**—(By a straight line is here meant a wire whose thickness is uniform, and infinitely small compared with its length ; the substance of the wire being also homogeneous throughout. When the centre of gravity of a line, straight or curved, is spoken of, the mass per unit length is always supposed to be uniform unless the contrary is specified. Similarly, the centre of gravity of a surface is found by supposing the thickness uniform and infinitely small, and the substance of the sheet homogeneous, so that the mass per unit area is the same over the whole surface. Solid figures are supposed to be of uniform substance throughout.)

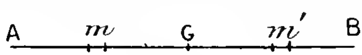


FIG. 88.

Let A B be a straight line, G its middle point (fig. 88) ; then, since every body has a centre of gravity, it might at once be inferred from symmetry that G is the point required, but a

more direct proof is easily given. Let A B be divided into a very large even number of equal parts (each part being consequently very small), and let m, m' be two of these parts which are equidistant from G and on opposite sides of it. Since m and m' have the same mass they have also the same weight, and the resultant of the two weights is a force acting vertically downwards through G. Similarly, all the parts of the straight line may be arranged in pairs similar to m and m', and the weights of each pair will have a resultant through G. Finally, compounding all these forces, the resultant acts through G, which is therefore the centre of gravity of the straight line A B.

184. **Centre of Gravity of a Parallelogram.**—Let the parallelogram A B C D (fig. 89) be divided by straight lines parallel to A B into a very large number of strips of equal width. These strips being excessively narrow, each may be treated like the straight line in the previous section; the centre of gravity of each strip will be at its middle point, and if each strip be replaced by a particle of equal mass at its centre of gravity, we shall have a number of equal particles disposed at equal intervals along the straight line E F, which bisects A B and C D; the centre of gravity G of these particles, and therefore also of the parallelogram A B C D, is at the middle point of E F. Similarly, G is the middle point of H K, which bisects A D and B C; it is also obviously the point of intersection of the diagonals A C and B D.

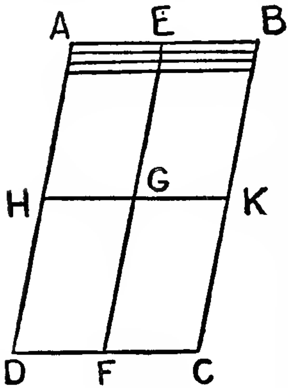


FIG. 89.

185. **Centre of Gravity of a Triangle.**—If the triangle ABC (fig. 90) be divided into very narrow strips by straight lines parallel to B C, each strip will have its centre of gravity at its middle point. But if D be the middle point of B C, A D bisects all the strips; it therefore contains the centre of gravity of each, and hence also the centre of gravity of the triangle.

Similarly, each of the 'bisectors,' B E, C F, contains the centre of gravity. Hence, *The three straight lines drawn from the angular points of a triangle to the middle points of the opposite sides meet in a single point, and this point is the centre of gravity of the triangle.*

That *the three 'bisectors' of a triangle are concurrent* is easily proved by pure geometry. For, let B E, C F be two bisectors of the triangle A B C (fig. 91), G their point of intersection, and D the middle point of B C; join A G, G D; then we have to show that the dotted line A G D is straight.

Since D, E, F are the middle points of the sides, we have, by Euclid I. 38, the triangle B G D equal in area to C G D

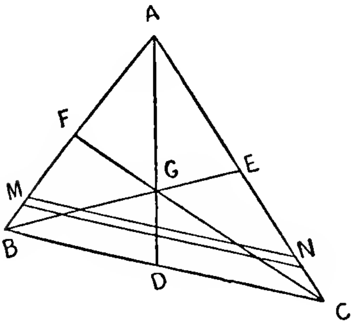


FIG. 90.

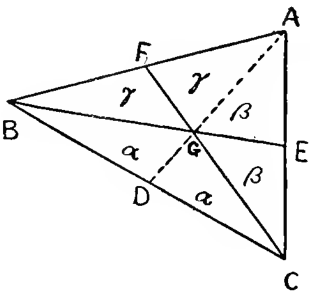


FIG. 91.

($=\alpha$, suppose); similarly, the area C G E $=$ A G E ($=\beta$, suppose), and A G F $=$ B G F ($=\gamma$, suppose). Similarly, the triangle B E C $=$ B E A $= \frac{1}{2}$A B C $=$ C F A $=$ C F B;

or, $2\alpha + \beta = 2\gamma + \beta = \frac{1}{2}(2\alpha + 2\beta + 2\gamma) = 2\beta + \gamma = 2\alpha + \gamma$,

whence $$\alpha = \beta = \gamma.$$

Therefore, $2\gamma + \alpha = 2\beta + \alpha$; that is, the triangle A B C is bisected by A G D, which is consequently a straight line. Thus the proposition is established.

Again, the triangle B G A $= 2\gamma = 2\alpha = 2$B D G, and the vertex B being common to the two triangles B G A, B D G, the base G A must be double of the base D G; or, D G $= \frac{1}{3}$D A.

Similarly, $EG = \frac{1}{3}EB$, $FG = \frac{1}{3}FC$. Hence, *The centre of gravity of a triangle lies on the straight line drawn from an angular point to bisect the opposite side, at a point one-third of the way from the point of bisection to the angular point.*

186. **The Centre of Gravity of the Circumference of a Circle** must be at its centre, as is evident from symmetry, or as follows: through the centre O (fig. 92) draw a very large number of diameters, the angle between consecutive diameters (such as A B, C D) being always very small. Then, the very small arcs A C, B D are equal to one another, and have equal mass and equal weight. Their centre of gravity is consequently at the point O, midway between them. Similarly, each two

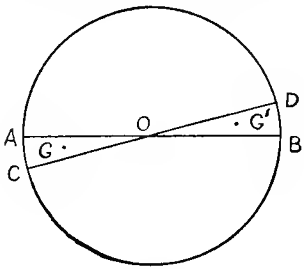


FIG. 92.

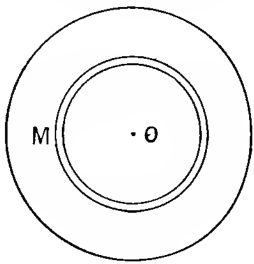


FIG. 93.

successive diameters will cut off from the circumference two very small arcs whose centre of gravity is at O. Thus O is the centre of gravity of the whole circumference.

187. **The Centre of Gravity of a Circle** is at its centre; for, if a very large number of circles be drawn concentric with the given circle and dividing its whole surface into very narrow circular bands, such as M (fig. 93), each of these bands, being of uniform thickness, may be treated as the circumference of a circle, and the centres of gravity of all the bands will be at the centre O of the given circle, which is therefore the centre of gravity of the circle itself.

We might also divide the area of the circle by a system of diameters like A B, C D (fig. 92). The centres of gravity, G, G′ of the triangles A O C, B O D lie on a diameter of the

circle, and each is distant from O by two-thirds of the radius. Thus O is the centre of gravity of two triangles A O C, B O D taken together, and similarly for each such pair of triangles.

188. **Centre of Gravity of a Sphere.**—Let any diameter of the sphere be taken, and let a system of planes be drawn perpendicular to it, dividing the sphere into a number of very thin uniform circular plates, each of which has its centre of gravity at its geometrical centre, that is, at the point where the plate is cut perpendicularly by the chosen diameter. Thus the centres of gravity of all the plates lie on this diameter, which therefore contains the centre of gravity of the whole sphere, and so, similarly, does any other diameter. The centre of gravity is therefore at the centre of the sphere, as might have been inferred from symmetry.

189. **The Centre of Gravity of a Spherical Surface** is also at its centre. A system of parallel planes, drawn as in the last section, will divide the surface into very narrow bands or zones, each of which is of uniform width, and may be treated like the circumference of a circle. Thus the diameter perpendicular to the system of planes contains the centre of gravity of each zone, and hence also of the whole surface. The same will be true of every diameter.

190. **Centre of Gravity of a Right Circular Cylinder.**—Draw a number of planes parallel to the base of the cylinder, dividing it into very thin uniform circular plates of equal thickness, and equal therefore in all respects. The centre of gravity of each plate is at its centre, that is, on the axis of the cylinder, and replacing each plate by an equal mass at its centre of gravity, we shall have a number of equal particles uniformly distributed along the axis of the cylinder. The centre of gravity of these particles, and therefore of the whole cylinder, is at the middle point of the axis.

191. The centre of gravity of a **cylindrical surface**, of a **rectangular solid**, of the **surface of a rectangular solid**, &c., are easily inferred from symmetry, or may be determined by methods similar to those already laid down.

192. **Centre of Gravity of a Triangular Pyramid.**—Let A B C D be a triangular pyramid (fig. 94), and let K be the

centre of gravity of the triangular face B C D. If a number of planes are drawn parallel to B C D, and dividing the whole pyramid into very thin plates (such as B′ C′ D′), the straight line A K will cut each of these plates at its centre of gravity (such as K′). For the triangles B C D, B′ C′ D′ will be similar, and the points K, K′ will be similarly situated with respect to them. Hence, the centre of gravity of the pyramid, which is made up of these plates, must lie on A K. Similarly, the centre of gravity must lie on each of the straight lines which join the angular points B, C, D to the centres of gravity of the opposite faces. This leads us to infer the geometrical truth, that the four straight lines drawn from the angular points of a triangular pyramid to the centres of gravity of the opposite faces all meet in a point.

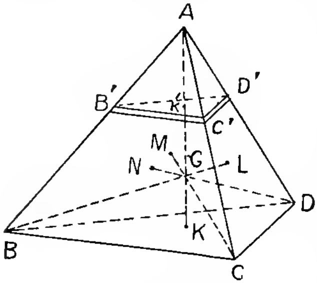


FIG. 94.

Taking a new figure (95) for the sake of clearness, let E be the middle point of C D; then L, the centre of gravity of the face A C D, lies on A E, and K, the centre of gravity of the face B C D, lies on B E. Join A K, B L, meeting at G. Then it has been shown that G is the centre of gravity of the pyramid. Now, since (§ 185) K E is one-third of B E, the triangle A K E is one-third of the triangle A B E, and, similarly, the triangle B L E is one-third of A B E. Subtracting the portion G K E L, which is common to the equal triangles A K E, B L E, we have A G L equal to B G K.

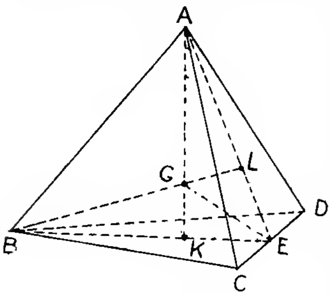


FIG. 95.

But A L = 2L E, and B K = 2K E; hence, twice the triangle L G E = A G L = B G K = twice K G E, and the

triangle A G E = A G L + L G E = 3E G K; and since the vertex E is common to the two triangles A G E, E G K, it follows that the base A G is three times the base G K, or that G K is equal to one-fourth of A K. Thus, *The centre of gravity of a triangular pyramid is on the line joining an angular point to the centre of gravity of the opposite face, at a point one-fourth of the way from this face to the angular point.*

193. **Centre of Gravity of a Polygonal Pyramid.**—Let A be the vertex, B C D E F the base of the pyramid, and O the centre of gravity of the base (fig. 96). Join A O, and let a number of planes be drawn parallel to the base of the pyramid, dividing its whole volume into very thin plates, such as B′ C′ D′ E′ F′, each of uniform thickness, and each similar to the base. The straight line A O will cut each plate at its centre of gravity, so that the centre of gravity of the whole pyramid must lie somewhere on A O. From O A cut off O G, equal to one-fourth of O A, and through G draw the plane B′ C′ D′ E′ F′, parallel to the base of the pyramid.

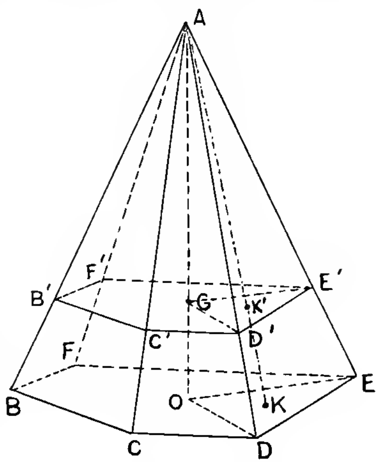


FIG. 96.

By joining O to each of the angular points B C D E F, the base may be divided into triangles, such as O D E, and the pyramid into triangular pyramids, such as A O D E; and if K is the centre of gravity of the triangle O D E, the centre of gravity of the pyramid A O D E will be at K′ on K A, such that K K′ = $\frac{1}{4}$K A; that is, K′ is the intersection of K A with the plane B′ D′ F′. Similarly, the centres of gravity of all the triangular pyramids lie in this plane, and so, therefore, does the centre of gravity of the whole pyramid. Hence, the point

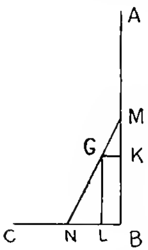
sought for is G, where the plane B′ D′ F′ is intersected by the straight line O A, and it is evident that, *The centre of gravity of any pyramid lies on the straight line joining the vertex to the centre of gravity of the base, at a point one-fourth of the way from the base to the vertex.*

194. **Centre of Gravity of a Cone.**—If the base instead of being a polygon, as in § 193, were wholly or partly bounded by a curved line or lines, the solid figure would be called a cone, and the rule just stated is also applicable to a cone, for by inscribing in the base a polygon with a sufficiently large number of sides, we may obtain a pyramid differing as little as we please from the given cone.

195. In some cases the centre of gravity is most easily found by dividing the line (or surface, or volume, or material body) into two or more parts, and finding the centre of gravity of each part separately. Supposing, then, that the whole mass of each part is collected at its centre of gravity, the problem reduces to that of finding the centre of gravity of a system of heavy particles.

Examples.

(1) To find the centre of gravity of a uniform wire A B C (fig. below), bent at a right angle, and having one limb, A B, twice the length of the other, B C.

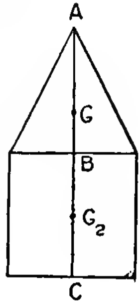


Bisect A B, B C in M and N respectively; then the centre of gravity of the whole wire will coincide with that of two particles, one at N and the other, of twice the mass, at M. This will be a point G on the straight line N M, such that N G = 2G M.

If the perpendiculars G K, G L be let fall on the limbs of the wire, it is evident, since G M = $\frac{1}{3}$N M, that G K = $\frac{1}{3}$N B = $\frac{1}{6}$C B, and K B = $\frac{2}{3}$M B = $\frac{1}{3}$A B. Thus the position of the centre of gravity is determined.

(2) The base of a square pyramid coincides with one face of a cube, and the height of the pyramid is equal to an edge of the cube; the density of the former body being 6 times that of the latter. Find the centre of gravity of the combination.

Let A be the vertex of the pyramid (fig. below), and let A B, its axis, be produced to meet the opposite face of the cube in C. The centre of gravity G of the pyramid will be on A B, G B being equal to one-fourth of A B; while the centre of gravity G_2 of the cube will be at the middle point of B C. Thus, B G_2 = 2G B. Again, the volume of the pyramid is one-third that of the cube, which stands on the same base and has an equal altitude; but since the pyramid has six times as great a density, its mass will be twice that of the cube, and it will also weigh twice as much.



Hence, the point B on G G_2, which is twice as far from G_2 as from G, is the centre of gravity of the combination.

196. When a given figure is most easily produced by removing certain portions from some simpler form, we may find the centre of gravity of each of these parts, and of the complete or unmutilated figure, as well as their relative weights (which will be known from their relative masses).

Now, when we commence with the complete figure, the effect of removing some definite part will be to *subtract a downward force*, equal to the weight of that part and acting through its centre of gravity, and this is the same thing as *adding an equal upward force* acting through the same point. We have, then, a downward force corresponding to the complete figure, and upward forces corresponding to the portions removed; the centre of these parallel forces is the centre of gravity of the given figure.

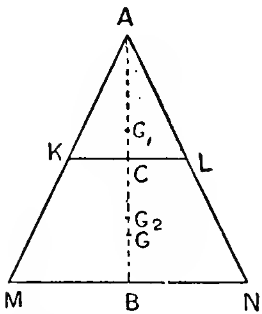
Examples.

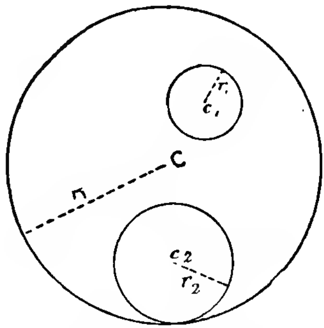
(1) A right circular cone is divided into two parts by a plane through the middle point of its axis parallel to its base. Find the centre of gravity of the larger part. (Figure overleaf.)

Let A M N be the given cone, A B its axis, whose middle point is C, and K L the intersecting plane. Let G_2 be the centre of gravity of the whole cone A M N, and G_1 that of the smaller cone A K L. Then $G_1 C = \frac{1}{4} A C$, and $C G_2 = G_2 B = \frac{1}{2} C B = \frac{1}{2} A C = 2 G_1 C$. Again, the solid figures A K L, A M N are similar to one

another, and their volumes are therefore as the cubes of their linear dimensions; that is, the cone A M N has 8 times the volume of A K L, and since we suppose the whole figure to be homogeneous, the masses as well as the weights of these cones will be in the same proportion.

Thus the point of application of the weight of the figure K M N L is to be found by compounding a downward force through G_2 with an upward force one-eighth as great through G_1. The resultant acts at a point G on $G_1 G_2$ produced, such that $G G_2 = \frac{1}{8} G G_1$, or $= \frac{1}{7} G_2 G_1$; and from this it easily follows that $B G = \frac{11}{56} . B A$. The required centre of gravity is therefore determined.





(2) A circular plate of uniform thickness and substance is pierced by a number of circular holes of given diameter and having given positions. Show how to find the centre of gravity.

Let C be the centre of the circular plate, R its radius; $c_1, c_2, \ldots$ the centres of the holes, and $r_1, r_2, \ldots$ their radii. The area of the uniform plate before piercing, and therefore also its weight, is proportional to R^2. The weights of the portions to be removed by piercing are respectively proportional to $r_1^2, r_2^2, \ldots$ Thus the centre of gravity of the pierced plate will be the centre of a system of parallel forces; one proportional to R^2 and acting downward at C; the others proportional to $r_1^2, r_2^2, \ldots$ and acting upward at $c_1, c_2, \ldots$ respectively.

197. **Properties of the Centre of Gravity.**—Let a body be freely suspended from some point A, other than G, its centre of gravity, so that the only movement possible is one of rotation

about the fixed point A (fig. 97). Let W be the weight of the body, acting vertically downward through G, and from A let fall the perpendicular A N on the line of action of W. Then the moment of the weight tending to turn the body about A is

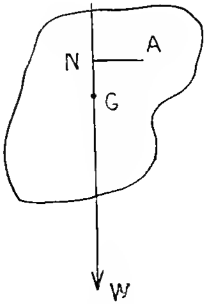


FIG. 97.

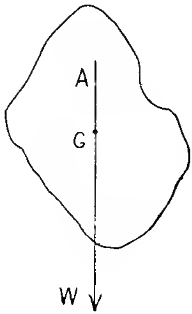


FIG. 98.

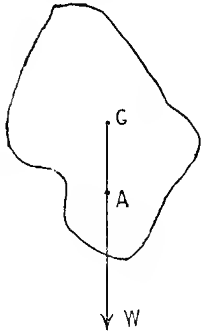


FIG. 99.

W · A N, and the support at A being supposed frictionless, there is nothing to oppose this moment, which will therefore produce rotation about the fixed point.

For equilibrium, then, the position of the body must be such that the moment W · A N is zero, and this will be the case when A N vanishes, that is, when G is on the vertical straight line through A.

There are thus two possible positions of equilibrium, in one of which the centre of gravity is vertically below, while in the other it is vertically above the point of support (figs. 98, 99).

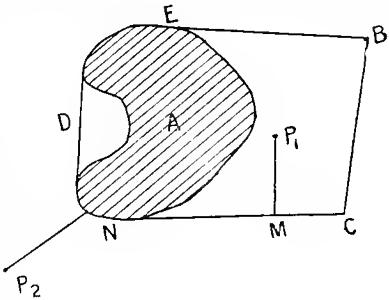


Fig. 100.

198. When a body rests upon a fixed plane, *the base is defined as the smallest figure which contains every point and surface of contact and is nowhere concave outwards.*

Thus, if the body is in contact with the plane at the points B, C, and over the surface A (fig. 100), the base on which it rests is defined as the area D N M C B E, which might in imagination be described by stretching a flexible string around A, B and C.

Having premised this definition, it will now be shown that, *When a body is placed on a plane, it will stand or fall according*

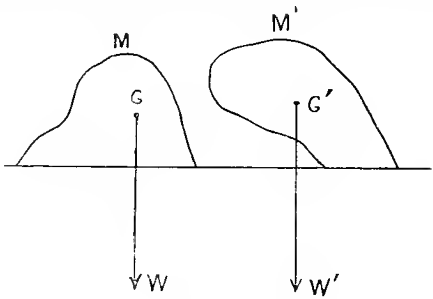


FIG. 101

as the vertical line through its centre of gravity falls within or without the base on which it rests. For it is obvious that, if the centre of gravity were vertically above P_1, the moment of the weight about any part of the base (such as the moment $W \cdot P_1 M$ about N C) would tend to produce such a rotation as would be opposed by the reaction of the plane. If, however, the centre of gravity is vertically above some point P_2 outside the base, the body will be tilted,

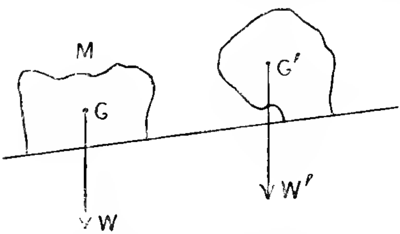


FIG. 102.

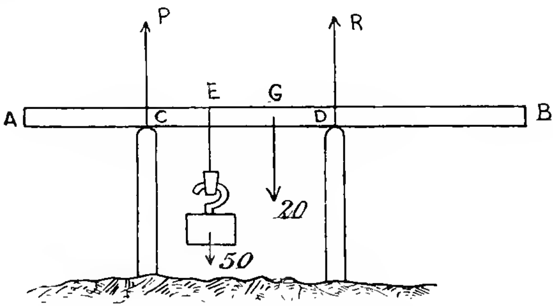
there being nothing to oppose $W \cdot P_2 N$, the moment of the weight about one point of the boundary. The two cases are illustrated by fig. 101. M will stand, but M′ will fall. In fig. 102 the bodies are represented as resting, or about to fall,

on an inclined plane, and it must here be supposed that there is friction sufficient to prevent sliding.

199. If a body stands on a rough horizontal plane, the vertical line through the centre of gravity intersecting the base at a point P, and if the plane is then gradually tilted, the point P will move towards one side of the base, and will finally reach the boundary. The body is now in a critical condition, so that the smallest impulse or additional inclination of the plane will cause it to overturn.

200. We shall now discuss some problems which illustrate the principle that the weight of a body is a force acting vertically downwards through its centre of gravity.

(1) A uniform beam, whose mass is twenty kilograms and length 4 metres, is supported horizontally on two props, 1 metre and 1·5 metre respectively from its ends A and B, and a mass of 50 kilograms is suspended from a point 1·5 metre from the end A. What forces are exerted on the beam by the two supports?



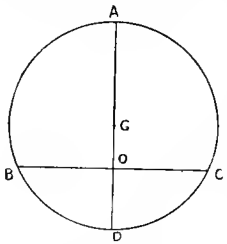
Let C and D be the points of support, P and Q the corresponding reactions, and E the point from which the mass of 50 kilograms is suspended. Then C E = E G = G D = ·5 metre. To find R, take moments about C; thus, $50 \cdot \frac{1}{2} + 20 \cdot 1 = R \cdot \frac{3}{2}$; or, R = 30 kilograms' weight; that is, R is a force of 30,000 g dynes, where g is the local value of the acceleration of gravity.

Similarly, P might be found by taking moments around D;

but its value now follows more easily from the condition that P + R = 50 + 20 kilograms' weight, whence P = 50 + 20 − 30 = 40 kilograms' weight, or 40,000 g dynes.

(2) A circular table 3 feet in diameter, and whose total mass is 40 lbs., is supported on three legs, inserted at equal intervals round its circumference. Find the greatest mass which may be placed at the edge of the table half-way between two legs without causing it to overturn.

Let A, B, C be the positions of the legs, and D the point where the mass is placed. Then the straight line A D passes through G, the centre of the circle, which we shall suppose to be vertically over the centre of gravity of the table. By symmetry it follows that G is the centre of gravity of the equilateral triangle A B C, so that O G $= \frac{1}{3}$ O A $= \frac{1}{2}$ G A $= \frac{1}{2}$ D G, or D O = O G. Now, the movement whose possibility is in question is one of rotation about a straight line parallel to B C, and lying vertically beneath it; that straight line, namely, which joins the feet of the legs B and C; and if forces W and P act downwards at G and D respectively, their moments about this straight line are W·O G and P·O D, which tend in opposite directions. Since D O = O G it follows that if W is greater than P, the former will predominate, the leg at A being pressed with some force against the ground; if W = P their moments just balance, while if P is greater than W the table will be overturned. The force acting through G is always the weight of 40 lbs., and thus 40 lbs. is the greatest mass which the table can support at the point D.



201. **Stability.**—*The equilibrium of a body is said to be stable when a very small displacement of the body calls forth such forces as tend to restore the former position; if, on the other hand, the forces due to a very small displacement have a tendency to produce further displacement, the equilibrium is called unstable; if no restoring or disturbing forces are brought into play, the small displacement will persist, and the equilibrium is called neutral.*

For example, let a body M be suspended from a point A, which is vertically above its centre of gravity, G (fig. 103, *a*). If the body be made to turn about A through a small angle, so that its centre of gravity moves from G to G′, its weight, W, will now have a moment W · A N about the point A, and the tendency of this moment will be to move the centre of gravity from G′ towards G—that is, to restore the body to its original position ; and the same will be true if the body receive any small rotation about the point A, excepting only when it is turned about the vertical axis A G. In this latter case, no restoring or disturbing moment is produced. *The equilibrium, then, is neutral as regards rotation about* A G, *and stable for all other rotations.*

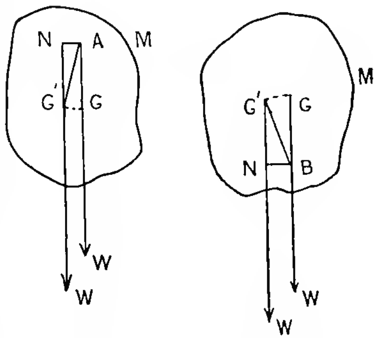


FIG. 103, *a*. FIG. 103, *b*.

Next, let a body M be supported at the fixed point B, which is vertically below G, and suppose that, by a small angular displacement about B, the centre of gravity is moved from G to G′ (fig. 103, *b*). The weight, W, has now a moment about B equal to W · B N, the tendency of which is to move the centre of gravity away from G ; that is, to displace the body still further from its original position. *Here, as before, the equilibrium is neutral as regards rotation around* B G, *but for all other rotations it is unstable.*

202. The general criterion for equilibrium and stability in such cases may be stated as follows :—*A heavy body supported by smooth rigid constraints will be in equilibrium if, for every kind of displacement permitted by the constraints, the centre of gravity* **begins to move** *in a horizontal or upward*[1] *direction ; and again, for those very small displacements which cause the centre of gravity to rise the equilibrium will be stable ; for those*

[1] Not necessarily *vertically* upward.

which cause the centre of gravity to fall, unstable; and for those which leave the centre of gravity at the same height, neutral.

An example of a smooth and rigid constraint is the inclined plane. Being *rigid*, the motion is entirely confined to the space on one side of the plane, and being *smooth* it offers not the slightest resistance to any motion which conforms to such restriction. The constraints on a rigid body may confine the motion of the centre of gravity to a *single line* (as when the body turns about a fixed axis along which it cannot slide); or to a *surface* (as when the body turns about a fixed point, or when it slides without rolling, or rolls without sliding on a fixed surface).

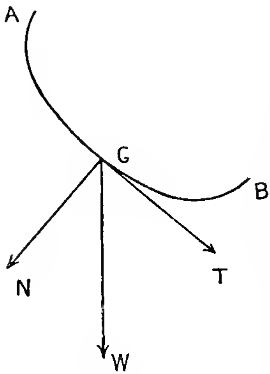


FIG. 104.

First of all then, let the constraints be such that the centre of gravity, G, can only move along the line A B (fig. 104). The weight of the body acts vertically downwards through G, and may in general be resolved into two components, one (T) along the tangent to A B at the point G; the other (N) perpendicular to T, and therefore normal to the curve A B. Since any motion of the centre of gravity perpendicular to A B is prevented by the constraints, the component, N, can produce no effect; on the other hand, motion along A B is entirely unrestricted, and the component, T, will accordingly displace the centre of gravity. It is evident, then, that for equilibrium, the component, T, must be zero; this is so when W is perpendicular to A B at the point G—that is, when the direction of A B at this point (as determined by its tangent) is horizontal. More generally speaking, the path along which G can move must have *no downward component*; otherwise the weight will have a positive component in the direction of a possible path, and will consequently produce displacement.

203. Fig. 105 illustrates some cases of equilibrium where

the centre of gravity is confined to a linear path. If the centre of gravity is free to move over a *surface*, then every straight line drawn touching the surface at the point occupied by the centre of gravity, will determine a possible direction in which the centre of gravity may begin to move, and from the foregoing it is evident that, for equilibrium, no one of these directions may have a downward component.

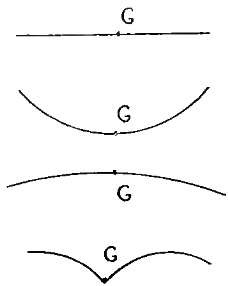


FIG. 105.

204. In the case of a rigid body which can turn about a fixed horizontal axis, the centre of gravity can only move in a vertical circle; and at the highest and lowest points of this circle, the tangent, which determines the direction of possible motion, is horizontal, and equilibrium will only obtain when the centre of gravity occupies one or other of these points.

If a body can turn freely about a fixed point, the centre of gravity, being always at the same distance from this point, must lie somewhere on the surface of a fixed sphere. At the highest and lowest points of this sphere, every tangent which can be drawn is horizontal, and there are thus two positions of the centre of gravity which are consistent with equilibrium.

Again (§ 198), when a body stands on a fixed plane, and when the vertical line through the centre of gravity falls within the base, the centre of gravity will be raised by every displacement which can be given to the body; while, if the vertical line falls outside the base, there are some directions of tilting which will lower the centre of gravity.

205. The second part of our enunciation (§ 202) concerns the stability of equilibrium; and it is evident that, if a very small displacement from equilibrium along any path has *raised* the centre of gravity from G to G′ (fig. 106), the *downward* direction along this path will be *towards* the former position; and since the weight of the body is acting downwards at the centre of gravity, it tends to restore equilibrium, and the condition is *stable* for the displacement in question. On the other

hand, if a very small displacement from equilibrium causes the centre of gravity to fall (fig. 107), the centre of gravity will be following a path whose *downward* direction tends *away from* the former position, and the weight will disturb the body still further from equilibrium. For such displacements, then, the condition is *unstable*. Finally, if a small displacement causes neither rise nor fall of the centre of gravity, the motion of this latter is entirely horizontal, and no restoring or disturbing forces are brought into play. The equilibrium is, therefore, *neutral* for the given displacement.

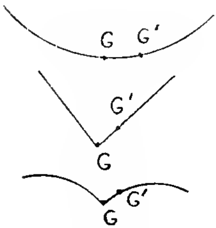


FIG. 106.

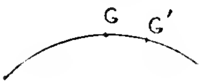


FIG. 107.

206. The stability of any condition of equilibrium must be decided by considering those displacements for which the stability is least. Thus, equilibrium is stable when, and only when, it is stable for every possible displacement ; it is neutral when, for every displacement, it is either stable or neutral ; and it is unstable if there is any displacement for which it is unstable.

207. The stability of a body suspended in equilibrium from a fixed point has already been considered (§ 201), and the results are evidently in accordance with the principles just laid down. As a further example we may take the case of a body which rests with a convex spherical surface, C N D, in contact with a horizontal plane (fig. 108). Let O be the centre of the spherical surface of which C N D is a portion, and G the centre of gravity of the body ; then the horizontal plane touches the surface at N, and O N is perpendicular to the plane. Since the only forces exerted on the body are its weight acting vertically downward through G, and the resistance of the plane acting vertically

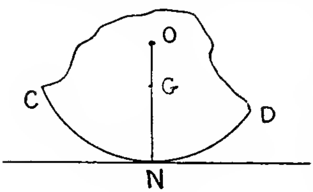


FIG. 108.

upward through N, it is necessary for equilibrium that their lines of action should overlie one another; that is, G must be vertically above N.

Again, it is evident that if the body receive a slight rolling displacement, the height of O above the plane will be unaltered, being still equal to the radius of the sphere; while, on the other hand, all points of the body between O and N will be *raised*. Thus, if the centre of gravity is at G, between O and N, the equilibrium will be stable; but if the centre of gravity is higher than O, it will be lowered by a slight rolling displacement, and the equilibrium will be unstable. Hence, *If a body, wholly or partly bounded by a spherical surface, rests with this surface in contact with a horizontal plane, it will be in equilibrium when its centre of gravity is vertically above the point of contact with the plane; and the equilibrium will be stable, unstable, or neutral, according as the centre of gravity is below, above, or coincident with the geometrical centre.*

208. Next consider the case of an egg-shaped body; a body, that is, whose figure may be generated by the revolution of an oval curve about its longer axis. Supposing the density uniform, the centre of gravity will (by symmetry) lie somewhere on the axis of revolution. If an egg be laid on its side on a

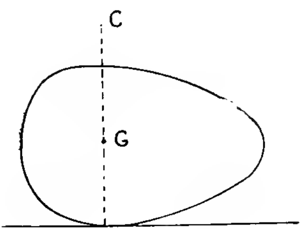


FIG. 109.

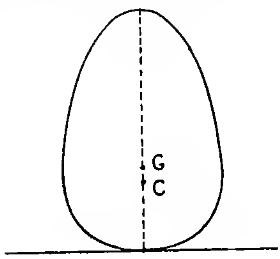


FIG. 110.

horizontal table, the equilibrium for one kind of rolling displacement will be neutral; for that, namely, which involves rotation about the longest axis, for in this case the centre of gravity remains at the same height above the plane. For

other rolling displacements, the equilibrium will be stable, for the centre of gravity G will be nearer to the plane than will the 'centre of curvature,' C, of the curve on which the egg rolls (fig. 109). If the egg were stood on end (fig. 110), the equilibrium would be unstable, for here the centre of gravity G lies above the centre of curvature, C, of the rolling surface.

209. **Balances.**—*A balance is an instrument for comparing the masses of bodies by the indirect method of comparing their weights at the same place on the earth's surface*; it is usually adjusted to equilibrium under the action of the weights of two bodies, and it then indicates some relation between their masses; for example, in

210. **The Common Balance** (fig. 111).—When the contents of the scale-pans have been so adjusted that the beam

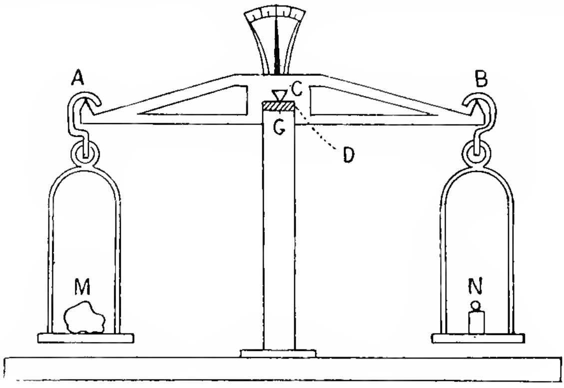


FIG. 111.

rests horizontally in equilibrium, the masses of these contents are equal to one another; that is, supposing the balance to be 'just.'

A B is the beam, and rigidly attached to it is the knife-edge C, a wedge-shaped body whose edge is turned downwards and has a horizontal direction perpendicular to the length of the beam. It rests on a fixed plane, D, of steel or agate, thus furnishing a fixed horizontal axis about which the beam can

oscillate. Similar knife-edges are fixed to the beam at A and B, having their edges turned upward, and from these are supported the scale-pans M and N. All the knife-edges are made as frictionless as possible, whatever friction there is being 'rolling friction,' and since the scale-pans and their contents are thus free to turn about horizontal axes, they will hang in equilibrium from the beam when their centres of gravity are vertically below these axes respectively, and the masses M and N of the scale-pans, with their respective contents, may be considered as collected at the corresponding knife-edges. Hence the forces acting downward on the beam are Mg at A, Ng at B, and Lg at G (L being the mass of the beam itself and G its centre of gravity). These must be in equilibrium with the resistance R of the horizontal plane D, acting vertically upward at the central knife-edge; the effects due to friction being neglected.

211. Suppose that when the beam is horizontal, G is vertically below C (fig. 112), and that the perpendicular distances of A and B from the straight line C G are a and b respec-

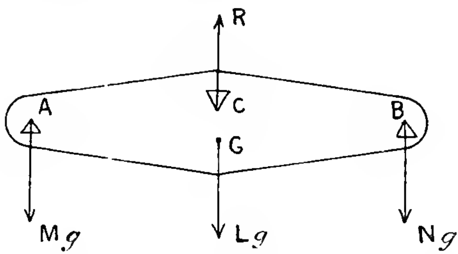


FIG. 112.

tively; then, taking moments about C, the condition for equilibrium in a horizontal position is that Mga shall be equal in magnitude to Ngb; that is,

$$Ma = Nb \quad\text{.........................}\quad \textbf{(44)}$$

If the balance is 'just,' $a = b$, and we must have $m = n$. In any case, suppose that when the pans are empty the beam rests horizontally, and that masses p and q can be placed in the

respective pans without disturbing the equilibrium position. The forces thus *added* are pg at A and qg at B, both acting vertically downwards, and since they balance one another, their moments about the central knife-edge must be equal and opposite; that is, $pga = qgb$ in magnitude, or

$$p/q = b/a \qquad \textbf{(45)}$$

that is, if the beam rests horizontally when the scale-pans are empty, all determinations of mass made with the balance will bear a constant ratio (a/b) to the true values; if the arms a and b are of equal length the ratio a/b is unity, and the balance is said to be *just.*

212. **Double Weighings.**—Let a body of mass p be placed in the scale-pan M (fig. 111), and let q_1 be the mass which must be placed in the other scale-pan to balance it, so that $pa = q_1 b$. Now let the body p be placed in the scale-pan N, and let q_2 be the mass required to balance it; here $q_2 a = pb$, and comparing this with the former equation, we see that $p^2 = q_1 q_2$, or

$$p = \sqrt{q_1 q_2} \qquad \textbf{(46)}$$

that is, *the true mass is the square-root of the product of the apparent masses obtained by weighing in opposite scale-pans.* In this way the measurement of a body's mass may be made independent of any inequality in the length of the arms, and at the same time the ratio a/b may be determined once for all; for

$$\frac{a}{b} = \frac{q_1}{p}, \text{ and } \frac{a}{b} = \frac{p}{q_2},$$

whence by multiplication

$$\frac{a}{b} = \sqrt{\frac{q_1}{q_2}} \qquad \textbf{(47)}$$

Practically the greatest accuracy is obtained by the *method of double-weighing.* The body whose mass is required is placed in one scale-pan, and is counterpoised by sand placed in the other; then, leaving the sand undisturbed, the given body is removed and replaced by known masses called 'weights,' which are adjusted by trial until equilibrium is restored. The mass

thus substituted must be equal to the mass required, for the weight of each has produced exactly the same effect when acting under the same circumstances.

So much, then, for the theoretical accuracy of the balance; but there are other qualities, such as sensitiveness and stability, which very largely determine the practical utility of the instrument. A balance is *sensitive* when it gives appreciable indications for very small differences of the masses contained in its scale-pans; it is *stable* when, after being displaced, it rapidly oscillates about its position of equilibrium, and the stability is greater the greater the moment of the restoring forces called forth by a given displacement.

213. **Sensitiveness.**—Let C, as before, be the axis about which the beam turns (fig. 113), F the centre of gravity of the beam and scale-pans, and H that of their contents, supposing these suspended masses to be collected at the corresponding knife-edges, in accordance with § 210. Also, let G be the centre of gravity of the whole movable system. Suppose, now, that a small additional mass m is placed in the left-hand scale-pan; H will thus be displaced to H′ (say) and G to G′, and the beam will be again in equilibrium when G′ is vertically under C, that is, after turning through an angle G′CG. If the sensitiveness is to be great, then for a given value of m, the angle G′ C G should be as great as possible; and to this end C G *must be made as small as possible*, and G′ G as large as possible. Let B g be the weight of the beam and scale-pans, acting at F, 2M g the weight of the contents of the pans acting at H; then

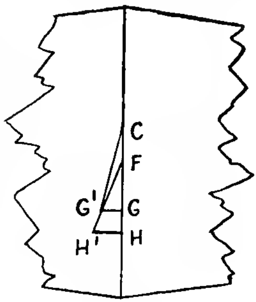


FIG. 113.

$$G\,F/H\,G = 2M\,/\,B,$$

$$\text{and } G'\,F/H'\,G' = (2\,M + m)/B = 2M/B \text{ very nearly,}$$

since we suppose m to be a *small* added mass. Thus the triangles F G G′, F H H′ are similar, and

$$G\,G'/H\,H' = G\,F/H\,F = 2M/(B + 2M)\ ;$$

or,
$$G\,G' = H\,H' \times \frac{2M}{B + 2M}.$$

Now H bisects the length of the beam, and H′ divides it in the ratio of M : M + *m*, so that H H′ *is evidently greater in proportion as the length of the beam is greater* ; also, *the fraction* 2M/(B + 2M) *is greater the smaller the mass* B *of the beam and scale-pans.* To secure, then, the greatest sensitiveness :

(1) The arms should be as long as possible.

(2) The beam and scale-pans should be as light as possible.

(3) The centre of gravity of the beam and appendages should be a very small distance from the axis of suspension.

The fulfilment of conditions (1) and (2) is limited by the need for rigidity in the beam, while (3) is not consistent with

214. **Stability** ; for, suppose that when the beam would rest horizontally it is displaced through an angle G C G′ (fig. 114), so that the centre of gravity of the oscillating system is now at G′. The moment tending to restore the equilibrium position is

$$(2M + B)\,g \cdot G'G\ ;$$

and for a given angular displacement G C G′, G G′ will be greater the greater G C. In practice, however, stability is not of so much account, provided the time of a single oscillation is not very great, for the eye can judge pretty nearly whether the beam swings equally to either side of the horizontal position, in which case it should come to rest horizontally. The observation is assisted by having a pointer attached to the beam, and whose free end passes in front of a fixed scale (fig. 111).

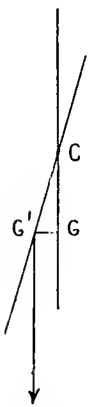


FIG. 114.

215. **The Common Steelyard** is represented in fig. 115. A E is a beam which can turn freely about a knife-edge, C. From a knife-edge A at one end is suspended a hook (or scale-pan), which supports the mass M to be measured

Another mass m whose value is known, can be adjusted to such a position D that the beam rests in a horizontal position.

The weight Bg of the beam acts at its centre of gravity G, and taking moments about C, the condition for equilibrium is

$$Mg \cdot CA + Bg \cdot CG + mg \cdot CD = 0;$$

or,
$$M \cdot AC = B \cdot CG + m\,CD,$$

(since $AC = -CA$). Of the quantities concerned in this equation, AC, CG, B and m are constant, while M and CD

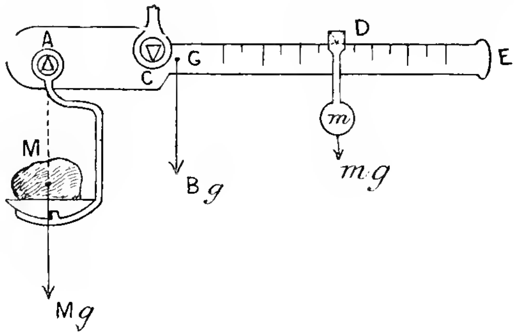


Fig. 115.

may vary in 'weighing' different bodies, and if M is increased by some quantity μ, m must be moved to some new position D′, to preserve equilibrium; where

$$(M + \mu) \cdot AC = B \cdot CG + m \cdot CD'.$$

Subtracting from this the preceding equation,

$$\mu \cdot AC = m \cdot DD';$$

or,
$$DD' = \mu \cdot \frac{AC}{m}.$$

Now, AC/m is a constant quantity, and it is therefore evident that a given displacement of m corresponds always to the same increment in the mass M, the instrument being therefore graduated by marking off successive equal lengths along the beam.

216. **The Danish Steelyard** (fig. 116).—In this instrument the fulcrum, C, is movable, the only other variable being the mass, M, to be measured, which is suspended from a knife-edge, A, at one end of the beam A E. At the end E, the chief mass of the beam is collected, so that the centre of gravity G is near

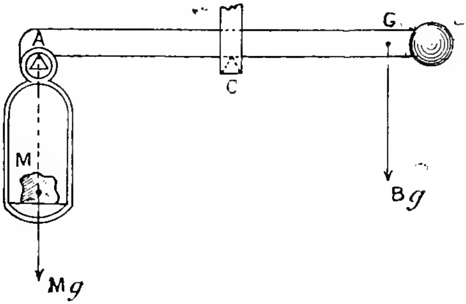


FIG. 116.

to this end. Thus the forces acting on the beam are its weight Bg, acting at G, the weight Mg, acting at A, and the resistance of the knife-edge, C. Taking moments about C, we must have for equilibrium,

$$M g \cdot A C + B g \cdot G C = 0 ;$$

or,

$$M \quad A C = B \cdot C G.$$

It is evident from inspection that, whatever be the values of M and B, there will be equilibrium for some position of C between A and E, and thus the *range* of the instrument has no theoretical limit.

EXAMPLES ON CHAPTER XIII.

(1) Masses proportional to 1, 3, 2, 4 respectively are placed at the consecutive corners of a square Find their centre of gravity

(2) Find the centre of gravity of three sides of a rectangle.

(3) Find the centre of gravity of seven edges of a cube.

(4) Find the centre of gravity of five faces of a cube.

(5) Find the centre of gravity of the perimeter of a triangle.

(6) Show that the centre of gravity of a triangle coincides with that of three equally heavy particles placed at its angular points.

(7) If the centre of gravity of a triangle coincides with that of its perimeter, show that the triangle is equilateral.

(8) Find the centre of gravity of a square, on one side of which a semicircle has been described.

(9) G is the centre of gravity of a triangle A B C ; find the centre of gravity of the figure A B G C A.

(10) On the radius of a given circle as diameter, a second circle is described. Find the centre of gravity of the figure formed by removing the second circle from the first.

(11) If a triangular board is supported at its three corners, show that each support bears one-third of the weight.

(12) The base of a right circular cone coincides with that of a right circular cylinder, and contains the centre of gravity of the whole solid figure so formed. Find the ratio of the altitude of the cone to that of the cylinder.

(13) A cylinder 20 centimetres high stands on a rough plane, and when the plane is tilted 30°, the cylinder is just on the point of falling over. What is the radius of the base?

(14) The centre of gravity of a cylinder is half-way between the axis and the curved surface, and the cylinder is placed with its axis horizontal on a plane inclined 30° to the horizon, on which it can roll but not slide. Find the position of the centre of gravity when there is equilibrium.

(15) A body whose mass is one kilogram, when placed in one pan of a false balance, has apparently a mass of 950 grams. What mass will it appear to have when weighed in the other pan?

(16) Show that the length of the graduations in the common steelyard is inversely proportional to the movable mass.

(17) When both pans of a balance are empty, it is found that the beam is not horizontal, but that when masses m and n are placed in the respective pans, the horizontal position is assumed, and, further, that when m is replaced by m' and n by

n', the beam still remains horizontal. Find the ratio of the arms of the balance.

(18) The centre of gravity of a Danish steelyard is half a metre from the point of suspension of the body to be weighed, and when this body has a mass of 1 kilogram, the fulcrum is 30 centimetres from the centre of gravity. Find the position of the fulcrum when a body of mass m kilograms is being weighed.

CHAPTER XIV

PROBLEMS IN STATICS

217. In this chapter we shall consider some problems on the equilibrium of bodies when acted upon by coplanar forces. If coplanar forces acting on a rigid body are in equilibrium, the following conditions are *necessarily* fulfilled :

(a) The sum of the components resolved in any direction in the plane must vanish.

(b) The sum of the moments about any point in the plane must vanish.

And these conditions will hold provided that,

(a') The sum of the components resolved in each of two given perpendicular directions in the plane vanishes

(b') The sum of the moments about a given point in the plane vanishes.

For the system of forces must either

(1) Be in equilibrium ; or,

(2) Be equivalent to a single finite resultant ; or

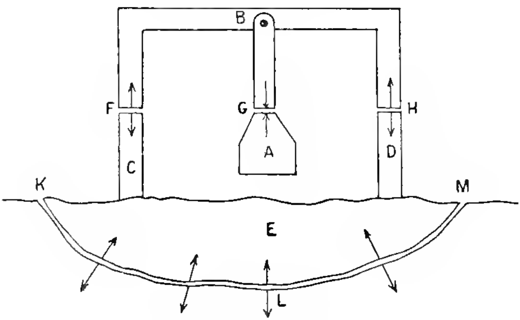
(3) Be equivalent to a couple.

If conditions (a') are fulfilled, there cannot be a finite resultant, for this resultant would have a component in any direction not perpendicular to it, that is, in either one or both of the given directions. The system, then, must either be in equilibrium, or must reduce to a couple. In the latter case the moment about every point in the plane would be the same, and (b') being fulfilled, this moment must necessarily be zero. Thus the conditions (a') and (b') are *sufficient* for the equilibrium of a rigid body acted upon by coplanar forces ; for although (a') contains but two, and (b') one condition, their

fulfilment implies that the infinitely numerous conditions in (*a*) and (*b*) are satisfied.

218. **Systems of Bodies.**—If a material system containing movable parts is in equilibrium under the action of forces applied to it, these forces must satisfy *amongst others* the same conditions as if the whole system were rigid; for if forces acting on a rigid body are not in equilibrium, then *à fortiori* they will not be in equilibrium when the parts of the body have greater freedom of motion. If the system is made up of a finite number of rigid bodies, it is necessary and sufficient that the forces acting on each of these bodies shall be of themselves in equilibrium; but we may of course consider any portion separately from the rest, such portion consisting of any body or bodies or parts of bodies in the system, and for equilibrium it is *necessary* though *not in general sufficient* that the impressed forces shall satisfy the conditions required by a rigid body.

219. For example, let a body A (fig. 117) be hung from the beams B, C, D, which are supported by the ground E; and let us imagine a surface arbitrarily drawn so as to divide the



. 117.

beams at F, G, H, while by another surface K L M a portion of clay E is separated from the rest of the earth; for the sake of clearness in the figure an actual separation is indicated, but the continuity of the bodies across these surfaces is supposed to be really unbroken.

The forces acting are the weights and the mutual reactions of the various bodies ; those which take place across the surfaces F, G, H, &c., being indicated by arrow-heads. Let a stand for the weight of A, b for the weight of B, and so on, and first of all consider the equilibrium of A. The only forces acting upon it are its own weight a acting vertically downwards, and an upward pull across the surface G which must be equal and opposite to a. Now take the woodwork B, which lies above the surface F, G, H ; it is in equilibrium under the action of the following forces: its own weight b acting downwards, a downward pull a at G due to the weight of the body A, and at F and H the resistances of the supports C and D, acting upwards. The resultant of these two resistances is therefore equal and opposite to the sum of the weights $b+a$. Finally, there is the portion made up of the supports C and D and the ground E ; the resultant action of the ground in contact with E must balance the weight $c+d+e$, together with the downward force $a+b$ exerted at F and H where the bodies A and B are supported; that is, the reaction across the surface K L M must balance the total weight $a+b+c+d+e$.

220. We have here imagined a division of the system which would not be profitable in practice. The three conditions for the equilibrium of a rigid body, in the case where the forces are coplanar, have been already given ; in the case where the forces may be any whatever, it may be shown that six conditions are necessary and sufficient for equilibrium, and it will not at all help us to find these if we suppose the body to be arbitrarily divided into a number of parts, whose equilibrium is then separately considered.

On the other hand, if a body with movable parts is treated as rigid, the relations obtained, though *necessary* for equilibrium, will *not* be *sufficient*. In practice those relations should be sought which will most easily lead to the equilibrium conditions in a convenient form, and to this end it is useful to remember the following rules :

(1) When a body is so constrained that it can only move parallel to a fixed direction, each of the impressed forces should be resolved into two components, acting respectively *parallel*

to this direction, and *perpendicular* to it. The latter components, being necessarily balanced by the reactions of the constraints, will be without effect, and only the former need be considered. In the case of a body which can only move over a plane or other surface, each force may be so resolved that one component is normal to the surface, and the other component tangential; the normal components being then left out of account.

(2) When one point or one straight line in a body is fixed, it is best to take moments about that point or straight line, for such forces as may pass through the point or the straight line will be without moment, and may therefore be neglected.

These rules are exemplified in the next three sections.

221. *A uniform ladder* A B, *inclined at an angle* θ *to the horizontal, rests with one end* B *on the ground, and the other end* A *against a smooth vertical wall* A C. *Find the least coefficient of friction between the ladder and the ground, consistent with equilibrium* (fig. 118).

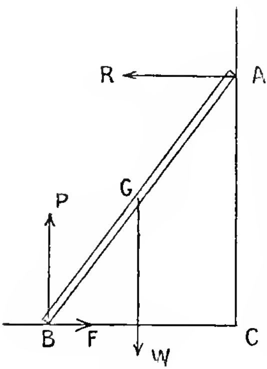


FIG. 118.

The forces acting on the ladder are: its weight W, acting vertically downwards through its centre of gravity G, which is at its middle point; the reaction R of the wall at A, which is perpendicular to the wall, (that is, horizontal); and the reaction of the ground at B, which may be split up into two components: P, vertical, and F, horizontal, the latter force being due to friction.

Resolving vertically and horizontally in the plane of the figure, it is evident that P is equal in magnitude to W, and R equal to F. To find the value of R or F, it is simplest to take moments about B: thus, R. C A + W$\cdot\frac{1}{2}$ B C must be zero, or R/W $= \frac{1}{2}$B C/C A in magnitude. Hence F/W $= \frac{1}{2}$ B C/C A, since F is equal in magnitude to R.

If the ladder were just on the point of sliding F/W would be equal to μ, and it follows therefore that no value of μ, less than $\frac{1}{2}$B C/C A—that is, less than $\frac{1}{2}\cot\theta$—is consistent with equilibrium.

222. *A wheel whose radius is* 20 *centimetres is mounted on a freely turning horizontal axle of radius* 2 *centimetres. A mass of* 200 *grams is attached to the circumference of the wheel, while round the axle is wound a flexible cord supporting a mass of* 1 *kilogram at its free end. Find the position of equilibrium of the system.*

Let fig. 119 represent an end-view of the machine, the point O corresponding to the axis of revolution, and A being the position of the mass of 1 kilogram when equilibrium obtains. Let N B be the vertical portion of the cord, and O N a horizontal radius of the axle; also draw A M A′ vertical, meeting the circumference of the wheel again in A′, O M being perpendicular to A M A′, and consequently horizontal.

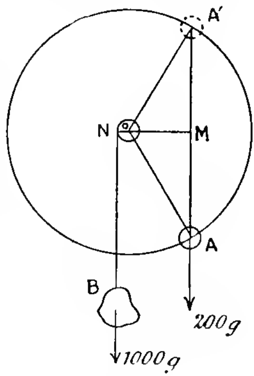


FIG. 119.

The only condition necessary for equilibrium is obtained by taking moments about the fixed axis; the weights of the bodies A and B are 200*g* and 1000*g* dynes respectively, and the moments of these forces about O are 200*g* · O M and 1000*g* · O N, which for equilibrium must be equal and opposite, so that as far as magnitude is concerned we have

$$\text{O M}/\text{O N} = 1000/200 = 5,$$

or
$$\text{O M} = 5 \cdot \text{O N} = 10 \text{ cm.} = \tfrac{1}{2}\,\text{O A}$$

Thus O M/O A (that is, sinO A M)$=\frac{1}{2}$, and the angle O A M$=30°$, the radius O A being inclined at this angle to the vertical.

It is evident that there will also be equilibrium when the lighter mass is at A′, the moment of its weight about the axis

being still equal to $200g \cdot$ O M, and the angle O A′M will also be equal to 30°.

It will be seen that the position A corresponds to *stable*, and A′ to *unstable* equilibrium. For when the smaller mass is at A, a small clockwise rotation of the wheel will diminish the moment of the weight about the axis, so that the weight of the larger mass will predominate and will tend to re-establish the equilibrium position; similarly, if the wheel is rotated in the contrary direction, the moment of the lesser weight, acting at A, will be increased, and will also tend to restore equilibrium. On the other hand when A′ is the position of the 200-gram mass, a small clockwise rotation, causing the weight of the greater mass to predominate, will lead to still further departure from the equilibrium position, and a similar result would follow from a small rotation in the contrary sense.

223. *Four rigid rods are hinged together at their extremities, so as to form a parallelogram* A B C D (fig. 120), *of which the side* A B *is fixed; and they are acted upon by forces lying in the plane of the parallelogram. What condition must be fulfilled that these forces may be in equilibrium?* (Fig. 120).

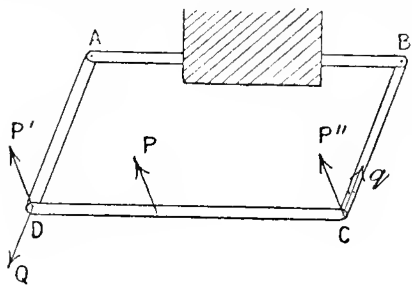


Fig. 120.

It may be remarked at the outset, that if one of the angles of the parallelogram is given, the whole configuration is completely determined, and accordingly we shall find that a single condition suffices for equilibrium. Let P be a force acting at a point in D C; then P may be replaced by a force P′ at D, which is equal and similarly directed to P, together with a couple whose moment is equal to that of P about the point D. By giving an appropriate magnitude to the constituents Q, q of this couple, they may be supposed to act at D and C in the directions A D, C B respectively; and the couple

is thus seen to be without effect. Again, the force P′, acting at the point D in the rigid bar A D, which can only turn about the fixed point A, may be replaced by any force acting on A D and having the same moment about A. In the same way, P might have been replaced by the equal and similarly-directed force P″ at C, and this again by any force acting on B C and having the same moment about B which P″ has about B, or P′ about A; and hence it follows immediately that any force acting on B C and having a given moment about B may be replaced by a force acting on A D and having an equal moment about A. Since A B is fixed, forces acting upon it will be without effect. The condition for equilibrium is therefore as follows: Let each of the forces acting on D C have its point of application transferred to D or to C; then the sum of the moments about A of all the forces acting on A D must be equal and opposite to the sum of the moments about B of all the forces acting on B C.

224. **Forces not in one Plane.**—We may now briefly refer to the method for compounding a number of forces acting at a point, and not all lying in one plane, the nature of the process being essentially the same as when the forces are coplanar. First of all, two of the forces may be selected and compounded by the parallelogram rule; the resultant of these two may then be compounded with another of the forces, and so on until we arrive at the resultant of the whole system.

225. **Three Forces** may, in general, be compounded by the construction known as the *parallelopiped of forces*. Let O A, O B, O C, drawn from the point O, represent the forces in direction and magnitude (fig. 121). Complete the parallelograms O A D B, O A E C, O C F B; and then complete the parallelograms A D H E, B D H F, C E H F. The solid figure contained by these six parallelograms is called a parallelopiped: its faces are parallel, two and two; and each of its twelve edges is parallel to three others.

Now the forces (=) O A and O C have a resultant (=) the diagonal O E of the parallelogram O A E C. Again, O E H B is a parallelogram, and the forces (=) O E, O B have a resultant (=) O H. Thus, O H represents in direction and magnitude the resultant of the forces (=) O A, O B, O C; and hence the

proposition: *If three forces be represented in direction and magnitude by three straight lines drawn* **from** *a point, and if a parallelopiped be constructed having these three straight lines for*

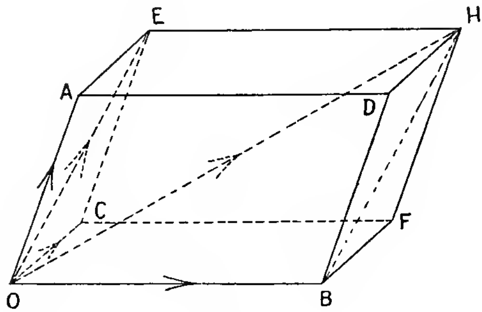


FIG. 121

adjacent edges, the resultant force will be represented in direction and magnitude by the diagonal of the parallelopiped drawn **from** *that point.*

226. It can be seen from the figure that the resultant might also have been found as follows :—*From* any point O draw O A, representing in direction and magnitude one of the forces ; *from* A draw A D (=) a second force, and *from* D draw D H, (=) the third. Then the straight line drawn *from* O *to* H represents the resultant force in direction and magnitude. The three forces may be compounded in any order, and we shall always arrive at the same point H, whether by the path O A, A D, D H, or O B, B F, F H, or O C, C E, E H, &c. It is also easy to see that a similar construction will hold good whatever the number of forces; and thus the polygon construction may be extended to any number of forces which are not coplanar.

227. The parallelopiped of forces is most useful in the case where the three given forces are mutually perpendicular, like those represented by O A, O B, O C (fig. 122), the parallelopiped being then called rectangular, since all those sides and faces which are not parallel are perpendicular to one another. E H

is thus perpendicular to the plane A O C, and, therefore, to the straight line O E, which lies in this plane; so that $OH^2 = OE^2 + EH^2 = OA^2 + AE^2 + EH^2 = OA^2 + OB^2 + OC^2$; that is, *the square of the resultant of three mutually perpendicular forces is equal to the sum of the squares of the forces.* Perfectly similar relations will of course hold good in the case of other

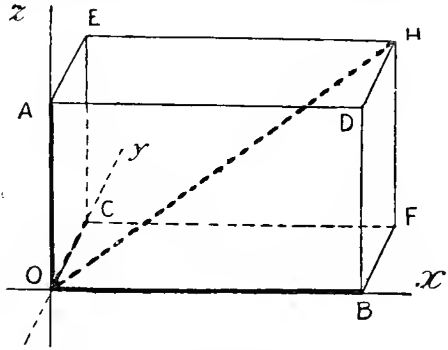


FIG. 122.

directed quantities, such as velocities, displacements, momenta, &c.; and it will further be evident that three mutually perpendicular axes, Ox, Oy, Oz, being chosen, any directed quantity O H may be resolved into three components in the directions of these axes; for if the perpendicular H F be drawn from H upon the plane Oxy, F H (or O A) is the component of O H in the direction Oz, and, similarly, the perpendiculars H E, H D on the planes Oyz, Ozx respectively, determine the components in the directions of Ox, Oy. The value found for any component may be positive, or it may be negative, or zero.

EXAMPLES ON CHAPTER XIV.

1. Two uniform planks, equal in all respects, are placed each with one end on the ground, while their other ends rest against one another; if the planks include an angle of 60°, what is the least value of the coefficient of friction between them and the ground which is consistent with equilibrium?

2. A wheel is prevented from rolling down a rough plane inclined at an angle θ to the horizon, by means of a string fastened to its rim, the string being tangential to the circumference of the wheel, and parallel to the inclined plane. Find the pull of the string.

3. What conditions are necessary for the equilibrium of a circular disc which lies on a smooth horizontal floor with its edge against a smooth vertical wall?

4. Show how to find geometrically the position of equilibrium of a heavy uniform beam, which rests with its ends on two given smooth inclined planes.

5. A gate is supported on two hinges, of which the lower bears all the weight, while the upper merely keeps the gate from falling away from its support; determine completely the reactions of the hinges.

6. A thin uniform heavy rod passes through a fixed ring 5 centimetres in diameter, and is thus supported in equilibrium at an inclination of 30° to the horizon. If the coefficient of friction is $\frac{1}{4}$, find the limiting position of the rod's centre of gravity which is consistent with equilibrium.

7. A rectangular box is held in any position, and a smooth homogeneous sphere occupies its lowest corner; show how to find the reactions between the sphere and the three sides of the box which support it.

8. Four strings are tied together at a single knot, and are pulled in different directions. Draw a diagram to show how the forces exerted by the several strings may be compared.

CHAPTER XV

MASS-CENTRE—IMPACT

228. **Centre of Mass.**—The centre of gravity of a body has been defined as a point through which the *weight* acts in every position of the body, and it has further been shown that, since the weight of each particle is proportional to its mass, the position of the centre of gravity may be inferred merely from the figure of the body, and the distribution of matter within that figure. The point thus determined may appropriately be called the *centre of mass*, or *mass-centre*, and we shall find that it has very important properties, quite apart from the consideration of weight. To find the mass-centre of a system of material particles, any two of the particles A_1, A_2 (fig. 123) are joined by a straight line, and on this line, between the particles, a point B_2 is taken whose distance from either is inversely proportional to the corresponding mass. The point thus found is called the centre of mass of the two particles, and if we suppose (for the purposes of this investigation) that these two particles are collected together at B_2, the centre of mass of this compound particle, taken together with another particle A_3 of the system, is the centre of mass of the three particles A_1, A_2, A_3, in their original positions. Thus we may proceed until we arrive at B_n, the centre of mass of the whole system, which will obviously be coincident with the centre of gravity previously found (§ 182).

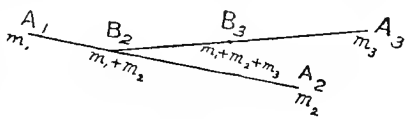


FIG. 123.

Similarly, in any material body the centre of gravity will coincide with the mass-centre, and we may find the mass-centre of a line, a surface, or a solid figure, on the supposition that the mass per unit length, per unit area, or per unit volume, is constant.

229. Now, in finding centres of gravity, it was assumed that the weight of each particle was proportional to its mass, and acted parallel to some fixed direction; that is (since $m g$ is the weight of a particle whose mass is m), that g is constant in direction and magnitude throughout the whole extent of the body. This assumption is very nearly true for bodies of moderate size at the earth's surface, but not so when very large bodies are concerned. For example, consider the earth itself, the magnitude of g, so far from being constant, changes from about 980 C. G. S. units at the surface down to zero at some point near the centre, while at different points of the earth's substance, g has every possible direction. The *centre of mass* can still of course be found by the method already given, but the term *centre of gravity* is no longer applicable; for, according to definition, it would be the point of application of the *weight* of the earth, that is, of the resultant force with which the earth attracts itself; and this resultant has of course no existence. On the other hand, the gravitative attraction of such a distant body as the sun is nearly uniform over the comparatively small space occupied by our planet, so that the resultant of the sun's attraction passes very nearly through the mass-centre of the earth.

230. **General Method for finding the Mass-Centre of any Material System.**—Let the particles of the system have masses m_1, m_2,... and let their perpendicular distances from some given plane be x_1, x_2,... ; M being equal to the sum of the masses and X the distance of their mass-centre from the same plane. Let K L and H G be any two straight lines at right angles to one another in the plane, and let forces $m_1 f$, $m_2 f$,... act on the respective particles, f being the same in each case, and all the forces being in the same direction as K L. Then we know from §§ 182, 228, that the resultant of these forces will be equal to $(m_1 + m_2 + \ldots) f$, that is, $\mathrm{M} f$, acting at the mass-centre of the system in the direction of K L.

Let A be the position of the particle m_1, and A N the perpendicular from A on the given plane. Let fall the perpendicular N S on H G, and also the perpendicular S T on the line of action of $m_1 f$. Then the moment of $m_1 f$ about the axis H G $= m_1 f \,.\, \mathrm{ST} = m_1 f \,.\, \mathrm{NA} = m_1 f \,.\, x_1$; whence it immediately follows that the algebraic sum of the moments about H G of the forces acting on all the particles is $m_1 f \,.\, x_1 + m_2 f \,.\, x_2 + \dots$; or $(m_1 x_1 + m_2 x_2 + \dots) f$; while the moment of their resultant about the same axis is $\mathrm{M}f \,.\, \mathrm{X}$. These moments being necessarily equal, we have,

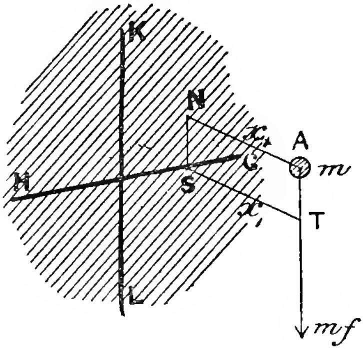


FIG. 124.

$$\mathrm{M\,X} = m_1 x_1 + m_2 x_2 + \dots;$$

$$\text{or} \quad \mathrm{X} = \frac{m_1 x_1 + m_2 x_2 + \dots}{\mathrm{M}} = \frac{m_1 x_1 + m_2 x_2 + \dots}{m_1 + m_2 + \dots} \quad \dots\dots(\mathbf{48})$$

If any other plane be thus considered, a similar result will, of course, be obtained. Thus, let there be three planes of reference, mutually at right angles (like the floor and two adjacent walls of a room, produced indefinitely): the position of a point will be completely determined when we know its distance, positive or negative, from each of the given planes; and if the mass-centre of a body be at distances X, Y, Z from the reference planes, we shall have, in addition to (48),

$$\mathrm{Y} = \frac{m_1 y_1 + m_2 y_2 + \dots}{m_1 + m_2 + \dots}; \text{ and } \mathrm{Z} = \frac{m_1 z_1 + m_2 z_2 + \dots}{m_1 + m_2 + \dots} \quad (\mathbf{48}a)$$

231. **Short Notation.**—It will be convenient to have a short notation for the resultant of a number of displacements or other directed quantities. We shall use the symbol $\nparallel$, placed between two directed quantities, to indicate that they are compounded by the parallelogram law. Thus if P, Q, S

are three forces, $P \oplus Q \oplus S$ will stand for their resultant, and we know from previous chapters that $P \oplus Q \oplus S = Q \oplus P \oplus S = P \oplus S \oplus Q$, and so on, in whatever order the forces may be compounded. Similarly we shall place the sign $\mapsto$ between two directed quantities to indicate that the first is to be compounded with a quantity equal and opposite to the second; thus, if A B, C D are two displacements, we have $AB \mapsto CD = AB \oplus (-CD) = AB \oplus DC$, and so on. If a number of directed quantities of the same kind are all in parallel directions, their resultant is equal to their algebraic sum, and $\oplus$ and $\mapsto$ may be replaced by + and − respectively.

232. **Theorem.**—*If any number of bodies are in motion, the velocity of their mass-centre is equal in direction and magnitude to the resultant of all their momenta, divided by the sum of all their masses.* For, let $x_1, x_2, \ldots$ be the distances of the mass-centres of the several bodies from some arbitrarily chosen plane, $m_1, m_2, \ldots$ their masses, and $u_1, u_2, \ldots$ the components of their velocities perpendicular to this plane (each of the x's and u's being taken with its proper sign). At the end of a short interval of time t, the distances of the mass-centres from the plane have become $(x_1 + u_1 t)$, $(x_2 + u_2 t), \ldots$; and if X denote the initial distance of the mass-centre from the plane, U the component of its velocity perpendicular to the plane, and M the sum of the masses, then, from equation (48),

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots};$$

and after the short interval t, this relation has become,

$$X + U t = \frac{m_1 (x_1 + u_1 t) + m_2 (x_2 + u_2 t) + \ldots}{m_1 + m_2 + \ldots}.$$

Subtracting from this the former equation, we obtain,

$$U t = \frac{m_1 u_1 t + m_2 u_2 t + \ldots}{m_1 + m_2 + \ldots}$$

or,

$$U = \frac{m_1 u_1 + m_2 u_2 + \ldots}{m_1 + m_2 + \ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots (49)$$

If we adopt a system of three mutually perpendicular reference planes, we shall have, in addition to equation (49),

$$\mathrm{V} = \frac{m_1 v_1 + m_2 v_2 + \dots}{m_1 + m_2 + \dots}\,;\ \mathrm{W} = \frac{m_1 w_1 + m_2 w_2 + \dots}{m_1 + m_2 + \dots} \quad \dots(49a)$$

where V, v_1, v_2, &c., are components of velocity perpendicular to the second plane, and W, w_1, w_2, &c., are components perpendicular to the third.

233. Now the velocity of the mass-centre of the system is the resultant of the three rectangular components U, V, W, and in the notation of § 231 is written $\mathrm{U} \dotplus \mathrm{V} \dotplus \mathrm{W}$; and this must be identical in direction and magnitude with the velocity found by compounding the right-hand sides of the three equations (49) and (49*a*); that is, we have,

$$\mathrm{U} \dotplus \mathrm{V} \dotplus \mathrm{W} = \frac{m_1 (u_1 \dotplus v_1 \dotplus w_1) \dotplus m_2 (u_2 \dotplus v_2 \dotplus w_2) \dotplus \dots}{m_1 + m_2 + \dots}$$

If ϕ_1, ϕ_2... are written for velocities of the several bodies, and Φ for that of the mass-centre of the system at the same instant, so that $\Phi = \mathrm{U} \dotplus \mathrm{V} \dotplus \mathrm{W}$, $\phi_1 = u_1 \dotplus v_1 \dotplus w_1$, &c., the last equation becomes,

$$\Phi = \frac{m_1 \phi_1 \dotplus m_2 \phi_2 \dotplus \dots}{m_1 + m_2 + \dots} \quad \dots\dots\dots\dots\dots\dots(50)$$

234. If no forces act on any body of the system, the momentum of each remains constant in direction and magnitude, and so, therefore, does the resultant momentum; hence the centre of mass of the system moves in a straight line with uniform velocity. But if the several bodies of the system are acted on by forces p_1, p_2,... which have given directions and magnitudes, they will have, respectively, the accelerations p_1/m_1, p_2/m_2,... which are in the same directions as the forces which produce them, and at the end of a short interval of time t, the momenta will have *increased* by the components $p_1 t$, $p_2 t$,... which are *not* necessarily in the same directions as the *initial* momenta, so that their values will then be $(m_1 \phi_1 \dotplus p_1 t)$, $(m_2 \phi_2 \dotplus p_2 t)$,... Let F be the acceleration of the mass-centre of the system, due to the action of the given forces. Then

during the very short time t, the velocity of the mass-centre has gained a component Ft, which must be compounded with Φ to give the velocity $\Phi \ddagger F\,t$ which the mass-centre has now attained. From (50) we have,

$$\Phi = \frac{m_1\,\phi_1 \ddagger m_2\,\phi_2 \ddagger \dots}{m_1 + m_2 + \dots};$$

and, at the end of the time t,

$$\Phi \ddagger F\,t = \frac{(m_1\,\phi_1 \ddagger p_1\,t) \ddagger (m_2\,\phi_2 \ddagger p_2\,t) \ddagger \dots}{m_1 + m_2 + \dots};$$

and, subtracting from this the former equation, we obtain,

$$F\,t = \frac{p_1\,t \ddagger p_2\,t \ddagger \dots}{m_1 + m_2 + \dots};$$

whence, dividing by t,

$$F = \frac{p_1 \ddagger p_2 \ddagger \dots}{m_1 + m_2 + \dots} \quad \dots\dots\dots\dots\dots\dots\dots\mathbf{(51)};$$

or, in words: *the mass-centre of the system has the same acceleration as if the whole mass were collected there, and the action of every force transferred to the same point.*

235. If the masses m_1, m_2,... have accelerations f_1, f_2,... respectively, the forces acting upon them will be $m_1 f_1$, $m_2 f_2$,... and the acceleration of the mass-centre is,

$$F = \frac{m_1 f_1 \ddagger m_2 f_2 \ddagger \dots}{m_1 + m_2 + \dots} \quad \dots\dots\dots\dots\dots\mathbf{(52)};$$

a result which might also have been deduced by considering the change produced during the short time, t, in the two sides of equation (50). Φ will have become $\Phi \ddagger F\,t$; ϕ_1, $\phi_1 \ddagger f_1\,t$, and so on; thus,

$$\Phi \ddagger F\,t = \frac{m_1\,(\phi_1 \ddagger f_1\,t) \ddagger m_2\,(\phi_2 \ddagger f_2\,t) \ddagger \dots}{m_1 + m_2 + \dots};$$

and subtracting (50) from this gives immediately (52) as before.

236. *The centre of mass of a rigid body has also this important property. that if the resultant of all the impressed forces*

passes through the centre of mass, it will have no tendency to produce rotation in the body. For, let M be the mass of the body, and P the resultant force (given in direction and magnitude) which acts at the mass-centre ; then the acceleration of this point will be P/M, which may, for shortness, be called F. We shall suppose that the body is without rotation previously to the application of the force P, and that during the interval of time considered (which may be as short as we please), P remains constant in direction and magnitude. Now P (which is equal to M F) may be replaced by a system of forces in its own direction, acting on the particles which compose the body, each particle being impelled by a force equal to the product of its mass by the acceleration F. But a particle, if isolated from the body and free to move under the action of such a force, would have an acceleration F. Thus every particle of the body, *if unconstrained* by the remaining particles, *would have the same acceleration* F, so that the particles would exactly retain their relative positions. The motion, therefore, will not be modified by supposing all the particles to be rigidly connected together, and the whole movement is one of pure translation. Now that the body has become rigid, the system of forces may be replaced by their resultant P, and we see that a force acting through the mass-centre of a free rigid body has no tendency to produce rotation.

Moreover, *the mass-centre is the only point which possesses this property*, for, if the resultant of all the forces does not pass through the mass-centre, it may be replaced by an equal force acting at this point together with a couple (§ 136). The force acting at the mass-centre has not any tendency to produce rotation ; but the couple has, its moment being equal to that of the original force about the centre of mass.

In § 228 mass-centre was defined as the result of an analytical operation, but in subsequent sections it has been shown that the point so found has important physical properties, from which we see the usefulness of the definition and the propriety of the name.

The term *centre of inertia* is sometimes used in the same sense.

237. **Example.** *A number of bodies are moving unconstrainedly under the action of their own weight; to show that their mass-centre moves in the same manner as a projectile.* Let the masses of the bodies be m_1, m_2,... ; then since each has the same acceleration g, directed vertically downwards, the acceleration of the mass-centre

$$=\frac{m_1 g + m_2 g + \dots}{m_1 + m_2 + \dots} = g\,;$$

there being no component of acceleration in any but the downward direction. Hence if the mass-centre has any horizontal velocity, that horizontal velocity will remain unaltered, and the motion will exactly resemble that of a projectile.

238. **Newton's Third Law of Motion** is thus enunciated: *To every action there is a corresponding reaction, equal in magnitude and opposite in direction.* The word *action* is here used in the sense of *force*, the *reaction* being an equal and oppositely directed force acting on some other body, or on another part of the same body. The two names are really interchangeable; whichever force is called the *action*, the other is to be called the *reaction.*

For example, let A and B be two bodies of equal mass M, (each, say, of the size of this planet) and placed some distance apart; A and B will attract one another, *and* A *will attract* B *exactly as much as* B *attracts* A; that is, if A experience a force P on account of B's attraction, B will experience a force $-$P on account of A's attraction;

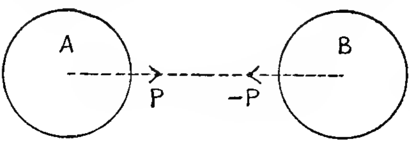


FIG. 125.

the acceleration of A will be P/M, and of B $-$ P/M, the changes of velocity which occur in the short time t being $+$ (P/M) t and $-$ (P/M) t, and the corresponding changes of momentum being $+$ P t and $-$ P t.

Next let the mass M_1 of A be greater than the mass M_2 of B. Newton's Third Law asserts that A will still attract B just

as much as B attracts A, although of course the mutual attraction (Q) is not now necessarily the same as P. The acceleration of A will be Q/M_1 and that of B, $-Q/M_2$, the increments of velocity in the short time t being $(Q/M_1)t$ and $-(Q/M_2)t$ respectively, and the corresponding changes of momentum being Qt and $-Qt$. Thus the *resultant momentum* of A and B has remained unchanged during the time t, and will continue unchanged, so long as the only forces acting on A and B are due to their mutual attraction. This result might have been inferred directly from the Third Law of Motion, since by equation (17) of p. 104 the resultant force acting upon any body is at each instant equal in direction and magnitude to its rate of change of momentum, and hence the two bodies A and B must have at each instant equal and opposite rates of change of momentum.

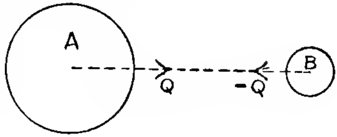


Fig. 126.

239. More generally, it is evident that if a material system be unacted on by any external force, so that for each force acting on any of the bodies there is an equal and oppositely directed force acting on some other part of the system, the resultant obtained by compounding all the forces of the system is zero, and consequently the mass-centre is without acceleration (§ 234). This is sometimes expressed by saying that *the internal reactions of a material system have no effect on the motion of its centre of mass.*

Thus an isolated material body or system of bodies cannot *as a whole* experience any force, and since the principle embodied in the Third Law of Motion is essential to the very nature of force, it may be as well to frame a definition which shall be in some measure based on this idea, thus: *Force is a mutual action which is exerted between two portions of matter, and which changes or tends to change their condition of relative rest or motion.*

240. Let us now return to the consideration of two bodies, and let one of these be the earth and the other a man standing on the earth's surface. Call the mass of the earth M, and that of the man m; M being taken to include the mass of all the

other people and objects which are materially connected with the earth. The earth then attracts the man with a force W, which is called his *weight*, and the man equally attracts the earth. If the man jumps into the air he imparts to his body some velocity (say v) which is directed away from the earth, and he has thus acquired a momentum mv. At the same time he must have generated in the earth an equal and opposite momentum $-mv$, which corresponds to a velocity $-mv/M$; a quantity quite inappreciable, since the earth has such an enormously greater mass. Of course all this time the earth is rotating on its axis, and revolving in its orbit round the sun, and is also subject to the disturbing influence of other people, not to mention rivers and oceans; but all the same the man, by his act of jumping, will add a new component to his momentum, and an equal and opposite component to the momentum of that material system whose mass we have called M. For by means of certain muscles he exerts a stress *between himself and the earth*, and the action of the stress is to drive these two bodies asunder. Had he attempted to jump upwards while falling freely from a height, he would have failed, for (neglecting the resistance of the air) all the forces he could exert would be between different parts of his own body, and the motion of his centre-mass could not thus be modified. While he is in the air he has a component of acceleration g due to the earth's attraction and directed towards the earth, while the earth's acceleration has a component of magnitude $(m/M)g$ directed towards the man, and due in turn to his attraction.

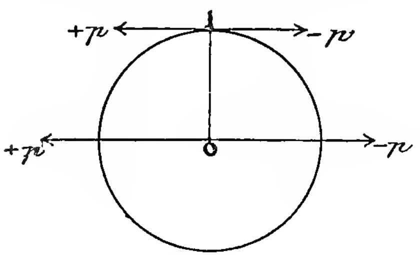


Fig. 127.

241. Now consider the case of a man who starts from rest, and walking along a horizontal road acquires a velocity v in a given direction. While he is gaining speed there is some force p acting upon him in the

direction of his motion, and an equal and opposite force ($-p$) acting upon the ground under his feet (fig. 127). As in § 136 we may add the forces $+p$ and $-p$ acting at the earth's mass-centre O, and we have then a force $-p$ acting at O, together with a couple pr in the plane of the diagram. Thus it will be seen that if a man starts walking round the earth he gives to the earth a component of rotation in the opposite direction; if he starts walking from west to east the earth turns a trifle more slowly; when he stops he restores to it that small component of angular velocity for which he is personally responsible; if he begins walking from east to west the earth will turn a trifle faster than if he had stood still.

242. An example frequently chosen to illustrate the Third Law of Motion is the kick of a gun or a cannon. If the cannon is mounted on a carriage whose wheels can run on rails with very little friction, the momentum given to the gun and carriage by the explosion will cause them to run along the rails for some distance before coming to rest. The rapid combustion of the powder causes an enormous pressure within the chamber of the gun, and if the construction of the gun is sufficiently strong, the chief effect will be to drive the gun and shot asunder with equal and opposite momenta. The more massive the gun and carriage the less will be their velocity (corresponding to a given momentum), and if the gun could be so fixed as to constitute together with the earth an ideally rigid body, the velocity generated in this body by the explosion would be quite insignificant, the *momentum* being equal and opposite to that of a cannon-shot.

The mass of the powder has here been neglected, and it has further been assumed that the forces exerted by the powder on the gun and the shot were the only forces acting. The latter assumption does not involve much error; for the force due to the explosion is so great, that *during the very short time for which it acts*, all other forces may be neglected in comparison. But during the flight of the shot, the chief force exerted upon it is due to the earth's attraction, while the force acting on the gun and carriage is mostly due to the friction of the wheels. Thus the mass-centre of gun, carriage, and shot will not remain all

the while at rest, *though no appreciable velocity was communicated to this point at the instant of the explosion.* Neglecting the resistance of the air, the mass-centre of the shot describes a parabola, and even should it burst into fragments while in mid-air, the mass-centre of all the fragments would continue undisturbed along the same parabolic path.

It may also be remarked that none of the processes described in these last three sections will have any influence on the motion of the earth's mass-centre, 'the earth' being of course taken to include the man, the cannon, the shot, and so on.

243. **Newton's Third Law of Motion** may also be put in the following form: *If any given portion of matter distributed in any manner be called* A, *and if any other portion of matter be called* B, *the resultant force exerted by* A *upon* B *is equal and opposite to the resultant force exerted by* B *upon* A. It must also be understood that action and reaction take place *along the same straight line.*

Let us now apply this principle to some examples.

(1) If a horse draws a stone along the ground by means of a rope, the stone is pulled forwards and the horse is pulled backwards with an equally great force; this was one of the examples given by Newton. The truth of the statement may

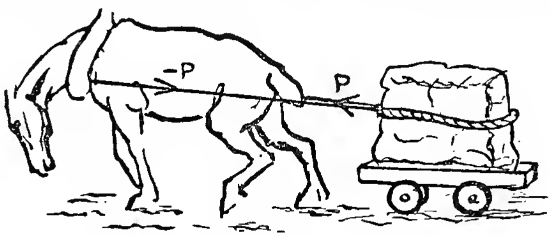


FIG. 128.

perhaps be rendered more evident from the following (fig. 128). Let M be the mass of the stone and m that of the horse, the mass of the rope being neglected, and let the motion be uniform. Then the pull P exerted on the stone by the rope must just be equal in magnitude to the friction between the stone and the ground, and P must also be the force with which the horse propels himself by pushing with his feet against the ground.

But since his velocity is constant, he experiences no force on the whole, and must therefore be pulled back by the rope with a force equal and opposite to P.

While the horse is still straining forward, let the rope be suddenly cut; the horse will now plunge forward, his rate of change of momentum being P; the stone on the other hand being retarded by the frictional force $-$ P, will have an equal and opposite rate of change of momentum. The effect of the rope was to equalise the motion: the stone was pulled forward, but to exactly the same extent the horse was pulled backward, and deprived of that acceleration which his exertions would otherwise have produced.

(2) Another instructive example is furnished by Atwood's Machine. We have seen (equation (15)) that if the moving masses are m and m' the pull of the string is $\frac{2mm'}{m+m'}g$; so that the weight of the body m being mg, the resultant force upon it is $mg - \frac{2mm'}{m+m'}g$ directed downwards, that is, $\frac{m(m-m')}{m+m'}g$.

Now let us imagine this body m to be divided by a horizontal plane into two parts, the mass of the upper part being μ, and that of the lower part $m-\mu$ (fig. 129). The mass $m-\mu$, being constrained to descend with a smaller acceleration than if it were falling freely, must experience an upward pull from its attachment to the mass μ, while on the other hand the mass μ falls with a greater acceleration than if $m-\mu$ were removed, and must therefore be subject to a downward force from its attachment to this latter mass.

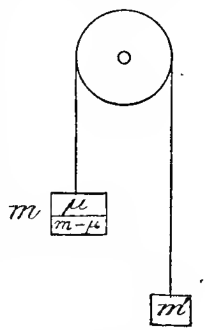


FIG. 129.

First take the mass μ. The forces acting upon it are (1) its own weight μg, (2) the pull of the string $-\frac{2mm'}{m+m'}g$ (written with the negative sign because it acts upwards on μ), and (3) a downward pull P due to the portion $m-\mu$. But since the mass μ has

a downward acceleration $\frac{m - m'}{m + m'} g$, the resultant force acting upon it is $\mu \frac{m - m'}{m + m'} g$, and equating this to the sum of (1), (2) and (3) we obtain

$$\mu g - \frac{2\, m\, m'}{m + m'} g + \mathrm{P} = \frac{(m - m')}{m + m'} \mu g,$$

$$\therefore \mathrm{P} = \frac{-\mu (m + m') + 2\, m\, m' + \mu (m - m')}{m + m'} g$$

$$= \frac{2\, (m - \mu)\, m'}{m + m'} g.$$

Next consider the mass $m - \mu$: the forces acting upon it are (1) its own weight $(m - \mu) g$, and (2) some force Q (whose value we shall find to be negative) due to its attachment to the portion μ. Again, since the mass $m - \mu$ has a downward acceleration $\frac{m - m'}{m + m'} g$, the resultant force acting upon it is $\frac{(m - m')\,(m - \mu)}{m + m'} g$, which must, of course, be equal to the sum of (1) and (2). Thus

$$(m - \mu) g + \mathrm{Q} = \frac{(m - m')\,(m - \mu)}{m + m'} g$$

and
$$\mathrm{Q} = \frac{(m - \mu)\,[(m - m') - (m + m')]}{m + m'} g$$

$$= - \frac{2\, (m - \mu)\, m'}{m + m'} g,$$

which shows that Q is an *upward* force, equal in magnitude and opposite in direction to P.

In whatever manner we may divide the mass m into two portions, the forces which these portions exert upon one another will be equal and opposite, and the same will of course be true for the mass m', as the student may prove without difficulty.

It has been found above that $\mathrm{P} = \frac{2\, (m - \mu)\, m'}{m + m'} g$; if this is written in the form

$$\mathrm{P} = \frac{m - \mu}{m} \cdot \left(\frac{2\, m\, m'}{m + m'}\right) g,$$

the quantity outside the bracket is a numerical fraction, being the ratio which the lower portion $m - \mu$ bears to the whole mass m; while the quantity within the bracket is equal to the pull of the string (T). If the suppositious surface A B is taken so as to make the upper portion μ very small, $(m - \mu)/m$ is very nearly unity, and P becomes nearly equal to T, the pull of the string; as A B is taken lower down a correspondingly smaller value will be found for P, and when $m - \mu$ is zero, P at the same time vanishes. This condition of things may be to some extent described by saying that the whole mass m is subject to stress; the stress across any section being proportional to the mass which lies below that section, and the stress across the highest layers being equal to the pull of the string. It has also been shown (§ 103) that when $m' = m$, the pull of the string is equal to the weight mg of m or of m', and that when m' is zero the pull of the string vanishes, so that m is falling freely. Since P is in this case zero throughout, it will be seen that a freely falling body experiences no internal stress from the action of gravity.

In §§ 93–96 no reference was made to the stressed condition of the moving masses, nor, as we now see, were such considerations needed; for, in accordance with Newton's Third Law, the acceleration of a body's mass-centre depends only on the forces impressed from without, and is independent of the mutual reactions between the parts of the body itself.

244. **Impulse.**—A force acting on a body during an interval of time gives to that body an impulse. The force may either be constant during the time considered, or it may vary in direction and magnitude. It has been seen (§ 98) that when a force acts on a free body, the rate of change of momentum is at each instant identical with the force, both in direction and magnitude, whatever be the mass of the body, and hence also it is evident that, during any given interval of time, the whole change of momentum depends only on the force (uniform or variable) and not on the mass of the body. *The impulse of a force (uniform or variable) which acts during a given time upon a free mass is equal in direction and magnitude to the whole change of momentum produced meanwhile.* Thus the **impulse** due to the con-

tinued action of a force is measured by **change of momentum,** just as the *force* itself is measured by *rate of change of momentum.*

If the force P is uniform during the time (t) considered, the change of momentum produced is P t, and this measures the impulse of the force during this time. For a **uniform** force, then, we have

$$\begin{aligned} \text{force} &= \text{rate of change of momentum}\,; \\ \textbf{impulse} &= \text{force} \times \text{time} \\ &= \text{rate of change of momentum} \times \text{time} \\ &= \textbf{change of momentum}\,; \end{aligned}$$

the equality including *direction* as well as magnitude.

Again, in § 116 it was stated that when a number of forces act simultaneously on a body during any given time, the same change of momentum is produced as if the forces had acted separately, each for an equal time, and this may now be expressed by saying that *the impulse during any time of a number of uniform or variable forces acting on a body is the resultant of all the impulses due to the separate forces.*

245. **Impulsive Forces.**—Consistently with our definition of impulse, all forces might equally well be called 'impulsive,' but the term has a special meaning in dynamics which agrees pretty closely with its ordinary acceptation : *a force is called impulsive when it produces an appreciable change of momentum in an inappreciably short time.*

Let us take an example ; a stone is thrown upwards into the air with a considerable velocity, which gradually diminishes owing to the action of gravity ; the weight of the stone is *not* called an impulsive force, because we are able to trace its gradual effect as the stone rises, comes for an instant to rest, and then falls to the ground. But suppose now that the stone falls on a bed of sand ; it will be brought to rest so quickly that we shall probably be unable to follow the changes in its velocity, while the distance to which it penetrates the sand will be but small. At the same time this distance may not be altogether negligible in comparison with the previous path of the stone, and we should not treat the change of momentum as impulsive if we

required more than a rough approximation to the truth. But let the sand be replaced by a block of granite; the change produced in the stone's momentum would be much more sudden, and the *distance* travelled by the stone *during the impulse* would be extremely small, being due, in fact, to the momentary compression of the hard colliding bodies.

When an appreciable change of momentum is produced in an extremely short interval of time, the force which acts during that interval must, of course, be correspondingly great; thus the stone while penetrating the sand exerts and experiences a force of much greater magnitude than its own weight, and it consequently sinks much farther than if it had been quietly laid on the surface. In the case of the granite the impact is of still shorter duration, and the force exerted may be sufficiently great to split the stone to fragments.

Impulsive forces differ only in degree, and not in kind, from forces which are not so named; no force is ever infinitely great, and a finite time is always required to produce a finite change of momentum; it will be seen, in fact, from the definition of impulsive forces in this section, that the application of the term depends ultimately on what we mean by an inappreciably small distance or an inappreciably short time.

246. If we know the momentum of a body both *immediately before* and *immediately after* the action of an impulsive force, we know what component of momentum has been added meanwhile, and moreover the interval of time will have been so extremely brief that forces of ordinary magnitude will have produced no appreciable effect. Thus *the whole change of momentum during this short interval* may without serious error be ascribed to the *impulsive force*, and will measure its *impulse* in direction and magnitude. Similarly, if we know the impulse of an impulsive force, we can sufficiently nearly trace its effect on the motion of a body, though its action is confined to so short a time that we are unable to follow its rapid variations, or to observe the very small path simultaneously described by the body. This is sometimes expressed by saying that *an impulsive force is completely known when we know the direction and magnitude of its impulse.*

To sum up then: an impulsive force is a very great force which acts during an inappreciably short time, and which thus produces an appreciable impulse; its time of action is so brief that

(1) the body acted upon has not meanwhile appreciably moved, and

(2) forces of ordinary magnitude have not appreciably affected the motion.

247. The following example will serve to illustrate the difference in effect between an impulsive force and a force of moderate magnitude whose impulse is the same (fig. 130). A B is a perfectly smooth fixed horizontal beam, and a small elastic ball projected obliquely upwards impinges on the beam at E; here it receives an impulse from the beam which causes it to rebound along the path E F. Since the beam is perfectly smooth, the force which it exerts on the ball is at each instant perpendicular to its surface; that is, vertical.

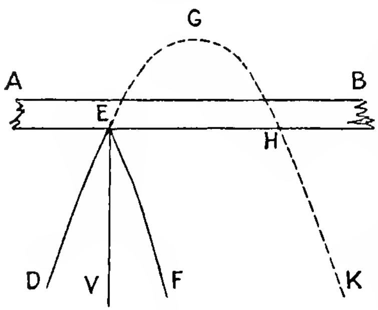


FIG. 130.

Hence the whole impulse is vertical, and if the ball has perfect elasticity (a property to be presently defined), the vertical component of its momentum will be exactly reversed, and the path E F subsequent to the rebound will be exactly similar to E D, but will lie on the other side of the vertical line E V. Had the beam A B been removed, the ball would have continued undisturbed along the parabolic path D E G H K, its horizontal velocity remaining unchanged, while at the point H, on the same horizontal level as E, the vertical component of its velocity (and therefore also of its momentum) would be equal and opposite to its value at E. Thus the path H K would be similar and similarly situated to the path E F, so that the ball, if undisturbed by the beam, would

have received from gravity during its passage from E to H *exactly the same impulse* which it actually receives from the collision.

The difference between the two cases is this: that gravity takes an appreciable time to generate the impulse, the ball meanwhile describing a finite path E G H, while the beam produces the same impulse nearly instantaneously, and before the ball can move appreciably from E; the weight of the ball producing no appreciable effect during the very short time of the rebound.

248. **The direct Collision of Bodies.** Let B and A (fig. 131) be two uniform spherical bodies in motion, and unacted on by any force, their masses being m and m' respectively, and their velocities u and u' (taken with their proper signs) being along the straight line which joins their centres. The velocity of B relatively to A is $u - u'$, and if

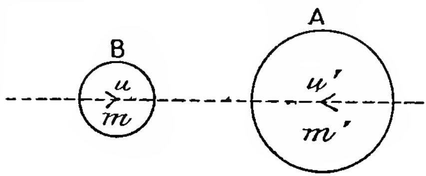


FIG. 131.

this be in the same direction as the distance B A, the bodies will ultimately collide, while from symmetry it is evident that after the collision the velocities will still be along the same straight line B A. The physical theory of such a collision may be roughly illustrated as follows: suppose there had been a spring S attached to one of the colliding bodies A (fig. 132), and suppose further that the body B, instead of striking directly against A, had encountered the free end of the spring. At each instant the degree of compression of the spring determines the force

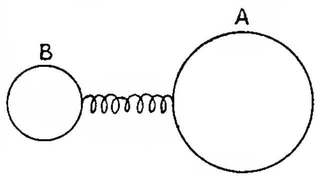


FIG. 132.

with which it tends to separate the two bodies, and these will have at each instant corresponding accelerations, which are so directed as to diminish the relative velocity. This velocity will

have been destroyed, and the bodies will be relatively at rest by the time that certain impulses $+ R$ and $- R$ have been received from the reaction of the spring.

Now if the working of the spring is accompanied by so much friction that it is unable to recover in any degree from the compression, the two bodies will remain relatively at rest, and in their present configuration will travel along together. This corresponds to the case of *inelastic* substances.

If, however, the spring could regain its former length, and could exert in expanding the same forces with which it resisted compression, it would separate the bodies with an impulse *equal* to and *in the same direction* as that by which the relative velocity was *destroyed*, and an equal and opposite relative velocity would consequently be generated. This corresponds to the case of *perfectly elastic* substances.

Finally, if the spring in expanding exerted less force than during compression, this force would produce during the rebound impulses $+ e R$ and $- e R$ where e is some quantity less than unity. Thus the relative momentum (and hence also the relative velocity) would have after impact $- e$ times the value which it had before impact.

When two hard bodies collide they slightly compress one another, and since they offer resistance to compression they are thus brought to relative rest, having received from one another equal and opposite impulses. If the substance of the bodies is inelastic it will not tend to recover itself when the compressing force ceases to act, and the two will not rebound from one another, but will travel along together. If, however, the bodies possess some degree of elasticity, and if the compression has been but small, they will return completely to their former figure, and in so doing will give an additional impulse to one another.

249. We may now consider more particularly the collision of the masses m and m' (fig. 131), whose momenta are at first $m u$ and $m' u'$. After the first part of the impact, in which the velocities are rendered equal, the momenta have become $m u + R$ and $m' u' - R$, where $+ R$ and $- R$ are the respective impulses which A has received from B and B from A. The

velocities, then, are $(m u + R) / m$, and $(m' u' - R) / m'$, and these are equal to one another ; that is

$$\frac{m u + R}{m} = \frac{m' u' - R}{m'}$$

whence, clearing of fractions,

$$m m' u + m' R - m m' u' + m R = 0,$$

or

$$R = \frac{m m' (u' - u)}{m + m'} ;$$

while each of the equal velocities $(m u + R) / m$ and $(m' u' - R) / m'$ is seen (on substituting for R) to be equal to

$$\frac{m u + m' u'}{m + m'} ;$$

this is in fact the velocity of the mass-centre of the two bodies, and remains unaltered by their mutual reactions. During the second part of the collision the bodies recover more or less perfectly their former figure, and receive from one another impulses $+ e R$ and $- e R$, where e is a numerical quantity less than unity, depending on the substances of the colliding bodies, and called the **coefficient of restitution.** The momenta have thus become $m u + (1 + e) R$ and $m' u' - (1 + e) R$, that is

$$m u + (1 + e) \frac{m m' (u' - u)}{m + m}$$

and

$$m' u' - (1 + e) \frac{m m' (u' - u)}{m + m'}$$

or

$$m \frac{(m - e m') u + (1 + e) m' u'}{m + m'}$$

and

$$m' \frac{(m' - e m) u' + (1 + e) m u}{m + m'}$$

Hence for the velocities v and v' after impact we have

$$v = \frac{(m - e m') u + (1 + e) m' u'}{m + m'},$$

$$v' = \frac{(m' - e m) u' + (1 + e) m u}{m + m'}$$

250. **Different Values of** e.—For some substances the coefficient e is very nearly unity; it can never be quite so great as unity, but in this ideal limiting case the elasticity is said to be *perfect*, the bodies being called perfectly elastic. With this value of e, the last equations become somewhat simpler; thus,

$$v = \frac{(m - m')u + 2m'u'}{m + m'},$$

$$v' = \frac{(m' - m)u' + 2mu}{m + m'}$$

If in addition the masses m and m' are equal, we have

$$v = u',\ v' = u$$

or in words, *if two equally massive and perfectly elastic spheres collide directly, they will interchange velocities.*

There are other substances again which are so 'inelastic' that after striking together they show no tendency to rebound from one another, the value of e being zero. In this case we have

$$v = v' = \frac{mu + m'u'}{m + m'}$$

the bodies after collision travelling together with a common velocity, which is necessarily identical with the velocity of their centre of mass.

251. The velocity of A relative to B is $u - u'$ before impact and after impact it is

$$\begin{aligned} v - v' &= \frac{(m - em')u + (1 + e)m'u'}{m + m'} \\ &\quad - \frac{(m' - em)u' + (1 + e)mu}{m + m'} \\ &= -e(u - u'), \end{aligned}$$

that is, $-e$ times the former value. This furnishes the most convenient definition of the coefficient of restitution, and it shows at once that when the bodies are 'perfectly elastic' the *relative* velocity is exactly reversed by the impact, and that when the bodies are inelastic no relative velocity will remain.

Experiment shows that for balls of cast iron the coefficient of restitution is about ·66, for balls of lead ·20, for balls of clay about ·17, for balls of glass ·94. When the colliding bodies are of different substances, the coefficient of restitution has generally a value intermediate between those of the bodies themselves, and approaching most nearly to that of the softer body. For example, the coefficient for two elm balls was found to be ·60, for brass balls ·36, and for a ball of elm and a ball of brass ·52.

252. In cases of direct collision between uniform spheres, the simplest relations to remember, and on the whole the best to apply to problems are: (1) the resultant momentum is unchanged by impact, (2) the relative velocity after impact has $-e$ times its value before impact.

Examples.

(1) Two 'perfectly elastic' spheres impinge directly, their masses being 8 and 12 grams, and their velocities respectively 15 and 10 centimetres per second in the same direction. Find the velocities v, v' of the spheres after impact.

The momentum of the system is throughout $8 \times 15 + 12 \times 10 = 240$ C.G.S. units of momentum; hence

$$8v + 12v' = 240;$$

and since the coefficient of restitution is unity

$$v - v' = -(15 - 10) = -5 \text{ cm. per sec.}$$

From these two equations we find

$$v = 9, \; v' = 14 \text{ cm. per sec.}$$

These velocities, being affected with the positive sign, are in the same direction as those before the collision.

(2) Two spheres, having collided directly, are now moving with velocities of 20 and -20, their respective masses being 6 and 8, and the coefficient of restitution $\frac{1}{2}$. What were their velocities (u, u') before impact?

The constancy of the resultant momentum gives

$$6u + 8u' = 6 \times 20 - 8 \times 20 = -40;$$

and the coefficient of restitution being $\frac{1}{2}$, we have

$$20 - (-20) = -\tfrac{1}{2}(u - u'),$$

or,

$$u - u' = -80.$$

From this and the former equation

$$u = -48\tfrac{4}{7},\quad u' = +31\tfrac{3}{7}.$$

(3) Two inelastic spheres A and B are moving in opposite directions at 6 metres per second and 8 metres per second respectively. They impinge directly, and then travel together with a velocity of 4 metres per second in the direction of A's previous motion. Find the ratio of their masses.

Let m be the mass of A, and m' that of B. Then the resultant momentum before impact is $6m - 8m'$, and after impact $4(m + m')$. Equating these values, $2m = 12m'$ or $m = 6m'$; that is, the more slowly travelling sphere has 6 times the mass of the other.

253. It has been assumed throughout that no forces act upon the colliding bodies except their mutual reactions, but if there be other forces, their influence during the very short time of the collision may in general be neglected, and the foregoing investigations will still apply, u and u' being the velocities *immediately before*, and v and v' *immediately after* the impact.

254. **Oblique Collision.**—If the velocities of both spheres are not along the straight line joining their centres, the collision is said to be *oblique*. The momentum of each sphere may then be resolved into two components, one along the line of centres, the other in some perpendicular direction. If the spheres are *smooth*, the force which they exert on one another will at each instant be along the line of centres, and so, therefore, will be the whole impulse given and received by each. The components of momentum in this direction will thus obey the same laws as if they were the only momenta; the components perpendicular to them remaining unchanged.

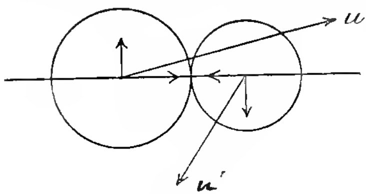


FIG. 133.

255. **Impact on a Fixed Plane.**—This is only a limiting case of the foregoing, the earth itself being one of the colliding masses. The collision will, of course, produce equal and

opposite changes of momentum in the two bodies, but owing to the enormous mass of the earth, its change of velocity will be inappreciable, and we may therefore regard it as fixed. Let A B, then, be the fixed plane (fig. 134), u the velocity of the sphere C just before impact, and v the velocity just after impact. We may resolve u into two components . h parallel to the plane A B, and k perpendicular to it, and supposing the surface perfectly smooth, the corresponding components of v will be h and $-ek$, where e is the coefficient of restitution. If the elasticity were perfect, e would be unity, and the components of v would be h and $-k$, the angle of rebound being then equal to the angle of incidence, and the magnitude of the velocity being unchanged by the collision.

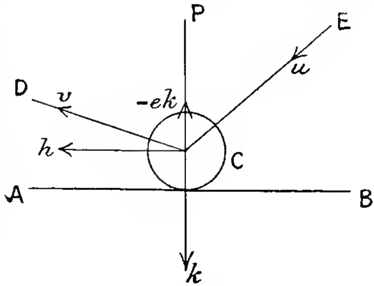


FIG. 134.

256. Thus a 'perfectly elastic' sphere, projected obliquely against a fixed horizontal plane, would describe a series of equal and similar parabolic arcs (fig. 135), but if, on the other hand,

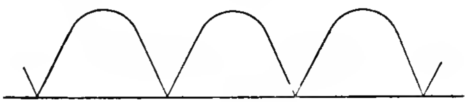


FIG. 135.

the sphere were 'inelastic,' the vertical component of its velocity would be destroyed by the impact, while the horizontal component h would remain unchanged, provided there were no friction ; thus the sphere after impact would glide along the plane with the velocity h.

257. If A B were a vertical surface (fig. 134) and if the motion of the sphere were constrained by a smooth horizontal plane, the velocity would remain constant in direction and magnitude

until the surface A B was encountered, when it would be suddenly changed, and would then once more remain constant.

Now let O X, O Y (fig. 136) be two smooth vertical planes, perpendicular to one another, and intersecting a smooth horizontal plane on which moves a uniform sphere. Let A B be the path of the sphere before the first rebound, the velocity being resolved into k perpendicular to O X, and h perpendicular to O Y. After the rebound at B the component h will be unchanged, while in place of k we shall have $-ek$. The sphere will now proceed along the path B C, and will rebound at C.

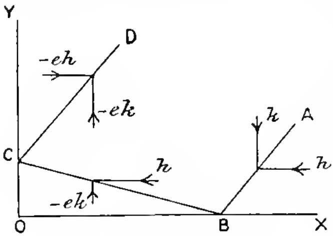


FIG. 136.

This second rebound will leave $-ek$ unchanged, but h will be changed to $-eh$. Hence each component of the present velocity is equal to $-e$ times the corresponding component of the initial velocity, and it is evident, therefore, that the whole velocity is now numerically equal to e times the initial velocity, and is exactly opposite in direction. The path C D, then, after the second rebound is parallel to the path A B.

Practically it is impossible to obtain bodies sufficiently smooth to render the friction negligible in an oblique collision; but the results here obtained may be taken as furnishing a rough approximation to the truth in certain cases, such as the impact of billiard balls with one another, or with the cushions of the table.

EXAMPLES ON CHAPTER XV.

(1) A uniform rod, whose mass is one kilogram, bears at one end a mass of two kilograms, and at the other end a mass of three kilograms. Through what point must an impulse act that the motion generated may be entirely one of translation?

(2) If one of the bodies of a system receives a given displacement, shew that the mass-centre of the whole system receives a

displacement in the same direction, and of m_1/M times the magnitude, where m_1 is the mass of the body in question, and M the mass of the whole system.

(3) Hence show that if this same body has at a given instant a velocity v and an acceleration f, the mass centre of the whole system will have in consequence a velocity $(m_1/\mathrm{M})\,v$ in the direction of v, and an acceleration $(m_1/\mathrm{M})\,f$ in the direction of f.

(4) Hence, by considering the velocity and acceleration of every body in the system, deduce the equations (50) of § 233 and (52) of § 235.

(5) If m and m' are the moving masses in Atwood's machine, show that the velocity of their mass-centre is

$$\left(\frac{m - m'}{m + m'}\right)^2 g.$$

(6) In the first system of three movable pulleys, each of the same mass, the 'weight' is that of a body suspended from the last pulley, and equal in mass to the sum of the masses of the pulleys. When the power moves through the unit of length, what displacement is thus given to the mass-centre of the system?

(7) A shell bursts into two portions whose masses are m_1 and m_2, and which move asunder with relative velocity v. Find the impulse received by either portion.

(8) Two equally massive spheres, whose coefficients of resilience are e and e' respectively, are dropped from the same height on a fixed horizontal plane; compare the impulses which they impart to the ground at their first impacts.

(9) There are a number of smooth straight tubes inclined at various angles to the vertical, and each containing a heavy particle. If all the particles begin sliding simultaneously from rest under the action of gravity, show that the path of their mass-centre is a straight line.

(10) A plank is laid on a *perfectly smooth* sheet of ice, and a man whose mass is three times that of the plank walks along it from end to end, the whole system starting from rest. Apply the third law of motion to compare the acceleration of the plank with that of the man at any instant.

(11) A sphere, dropped from a height upon a fixed horizontal plane, rebounds to one half the original height. Find the coefficient of restitution.

(12) A sphere A collides with another sphere B at rest; show that the velocity of B after impact can never be so great as twice the velocity of A before impact, and that this limiting value is most nearly approached when the coefficient of restitution is unity, and A very large compared with B.

CHAPTER XVI

ENERGY

258. **Work.**—When a body acted upon by a force is displaced in the direction of that force, the force is said to *do work*, and the work done is measured by the product of the force and the displacement; if the body were displaced in the opposite direction, work would be done *against the force*, while no work is done if the displacement is perpendicular to the force. Thus the weight of a mass m is a force $m g$ acting vertically downwards, and if the body descends through h centimetres the work done *by* its weight is $m g h$; if the body rises through h centimetres work $m g h$ is done *against* its weight, that is, its weight does work equal to $-m g h$. For horizontal displacements there will be no work done, either by or against the weight, while an oblique displacement may be resolved into two components of which one is vertical and the other horizontal; the product of the weight and the vertical component measures the work done *by* the weight, the product being positive when the displacement is downwards, and negative in the contrary cases.

259. **Unit of Work.**—It is evident from these definitions that the unit of work we have adopted is *the work done by the unit force when its point of application moves through the unit length in its own direction.* Thus the unit of force in the C.G.S. System being the dyne, the unit of work will be done by a force of one dyne when displaced one centimetre in its own direction; this unit of work is called an **erg**, and the work which must be expended to raise a mass of m grams through a vertical height of h centimetres is $m g h$ ergs.

In the British absolute system of units, the unit of work is performed by a force of one poundal acting through a displace-

ment of one foot in its own direction. This unit is called a **foot-poundal.**

In the so-called gravitation system of units, the unit of work is the work done by *the weight of a certain mass* acting through a displacement in its own direction equal to the unit of length. Thus to raise a pound of matter through a vertical height of one foot requires the expenditure of one **foot-pound** of work; but as we have seen that the weight of a given mass varies with the locality, the foot-pound is *not*, like the erg or the foot-poundal, a definite and invariable unit of work.

260. **Kinetic and Potential Energy.**—*The energy of a material system is its capacity for doing work.* Thus if a stone of mass m has been raised h centimetres above the ground, it is capable in its descent of doing $m g h$ ergs of work, and the system which consists of the earth and the stone possesses an *additional* $m g h$ ergs of energy on account of the relative positions of these mutually attracting bodies. The energy which a system possesses in virtue of the relative positions of its parts is called *potential energy.* If the parts of a system are moving relatively to one another, there will also be energy due to the motion, and this is called *kinetic energy.* For example, if a stone be allowed to fall from a height, the earth and the stone will acquire a velocity relatively to one another, and if their relative velocity be destroyed by an opposing force, work will be done against this force during the operation.

261. **Relation between Energy, Mass, and Velocity.**—Let a free mass m, starting from rest, be acted on by a constant force P during the time t, the distance described in this time being s. Then since the acceleration is P/m we have

$$s = \tfrac{1}{2} \cdot \frac{\mathrm{P}}{m} \cdot t^2$$

and the velocity acquired

$$v = \frac{\mathrm{P}}{m} \cdot t \ \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots \quad (\alpha)$$

The work done by the force P through the displacement s in its own direction is

$$\mathrm{P}\, s = \tfrac{1}{2}\, \frac{\mathrm{P}^2}{m}\, t^2 = \tfrac{1}{2}\, m \left(\frac{\mathrm{P}}{m} \cdot t\right)^2 = \tfrac{1}{2}\, m\, v^2 \text{ (from } (\alpha)\text{)} \ldots\ldots \ (\beta).$$

Let the force P now cease to act: the mass will continue to move with the uniform velocity v, and if at any time it meets with a directly opposing force Q, it will do work against this force before it is brought to rest. The acceleration produced in the mass m by the force Q is Q/m, and the time t taken to *destroy* the velocity v is $-v\,m/Q$ (which is a positive interval of time, since we suppose Q to be oppositely directed to v). The displacement described is $\frac{1}{2}v\,t' = -\frac{1}{2}v^2\,m/Q$, and the work done *by* Q through this displacement is

$$-\tfrac{1}{2}\cdot\frac{v^2\,m}{Q}\cdot Q = -\tfrac{1}{2}\,m\,v^2\,;$$

that is, work $+\,m\,v^2$ is done *by* the mass m *against* the force Q.

Thus the work which must be expended to give to the mass m the velocity v is $\frac{1}{2}\,m\,v^2$, and $\frac{1}{2}\,m\,v^2$ is also the measure of the work which the mass will do before it is brought to rest; it is the capacity for work which the mass possesses in virtue of its motion, and is therefore appropriately called the kinetic energy of the body; we may write, then

$$\textbf{kinetic energy} = \tfrac{1}{2}\,m\,v^2 \;\;\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

262. **Energy of Relative Motion.**—When we spoke of bringing a moving mass to rest, we probably meant bringing it to test relatively to the earth's surface. Had the motion of the body relatively to the sun been considered, the kinetic energy thus estimated would in general have been very different, and since all motion is essentially relative, we obviously cannot measure the *kinetic energy* of a body until we decide on some standard of position relatively to which *velocities* are reckoned. We may therefore speak with more propriety of *the energy of the relative motion of two or more bodies.*

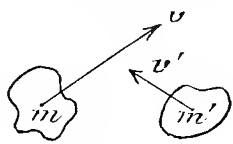


Fig. 137.

Thus let m and m' (fig. 137) be two moving masses, whose velocities (relative to some common standard) are respectively v and v'. The velocity of their mass-centre is

$$\frac{m\,v + m'\,v'}{m + m'}\,;$$

and their respective velocities relative to the mass-centre are

$$v \mapsto \frac{m\,v \ddagger m'\,v'}{m + m'} \text{ and } v' \mapsto \frac{m\,v \ddagger m'\,v'}{m + m'},$$

that is
$$\frac{m'\,(v \mapsto v')}{m + m'} \text{ and } \frac{m\,(v' \mapsto v)}{m + m'}.$$

The momenta
$$\frac{m\,m'\,(v \mapsto v')}{m + m'} \text{ and } \frac{m'\,m\,(v' \mapsto v)}{m + m'}$$

are of course equal and opposite—call then $+\mathrm{M}$ and $-\mathrm{M}$. Now let the respective forces $-\mathrm{M}/t$ and $+\mathrm{M}/t$ be exerted upon one another by the two bodies during the time t; the equal and opposite impulses $-\mathrm{M}$, $+\mathrm{M}$, of the two forces will not affect the motion of the mass-centre, but relatively to this point each body will be reduced to rest; that is, the velocity of each body will be $(m\,v \ddagger m'\,v')/(m + m')$. Again, during the time t the average velocity of the body m was

$$\tfrac{1}{2}\left(v \ddagger \frac{m\,v \ddagger m'\,v'}{m + m'}\right),$$

and the space described was

$$\tfrac{1}{2}\left(v \ddagger \frac{m\,v \ddagger m'\,v'}{m + m'}\right)t.$$

At the same time the space described by the body m' was

$$\tfrac{1}{2}\left(v' \ddagger \frac{m\,v \ddagger m'\,v'}{m + m'}\right)t.$$

And hence the displacement of the force $-\mathrm{M}/t$ exceeds that of the force $+\mathrm{M}\,t$ by

$$\tfrac{1}{2}\,(v \mapsto v')\,t\,;$$

this excess, either positive or negative, being in the direction of $v \mapsto v'$, that is of M and of M/t. Thus on the whole work done *against* the forces

$$= \tfrac{1}{2}\cdot\frac{\mathrm{M}}{t}\cdot(v \mapsto v')\,t = \tfrac{1}{2}\,\frac{m\,m'\,(v \mapsto v')^2}{m + m'}\,;\qquad\text{(54)}$$

this is therefore the expression for the **internal kinetic**

energy *of a system consisting of two masses m and m', whose relative velocity is $v \leftrightarrow v'$* ; it will be found to be identical with

$$\tfrac{1}{2} m \left[\frac{m'(v \leftrightarrow v')}{m+m'}\right]^2 + \tfrac{1}{2} m' \left[\frac{m(v' \leftrightarrow v)}{m+m'}\right]^2 \ldots\ldots (54\ a)$$

the quantities within square brackets being the velocities of the respective bodies relatively to the mass-centre of the system.

263. After this energy has taken some other form by doing work against the mutual reactions of the system, each body will have the velocity $(m v \ddagger m' v')/(m + m')$, which throughout is the velocity of the mass-centre. Thus the kinetic energy remaining is

$$\tfrac{1}{2}(m+m')\left(\frac{m v \ddagger m' v'}{m+m'}\right)^2 \text{ or } \tfrac{1}{2}\frac{(m v \ddagger m' v')^2}{m+m'} \ldots\ldots (55)$$

264. **Total Kinetic Energy.**—If we add together the expressions (54) and (55) we obtain

energy of relative motion + energy of motion of mass-centre

$$= \tfrac{1}{2}\frac{m m'(v \leftrightarrow v')^2 + (m v \ddagger m' v')^2}{m+m'}$$

$$= \tfrac{1}{2}\frac{m m' v^2 + m m' v'^2 - 2 m m' v v' + m^2 v^2 + m'^2 v'^2 + 2 m m' v v'}{m+m'}$$

$$= \tfrac{1}{2}\frac{(m^2 + m m') v^2}{m+m'} + \tfrac{1}{2}\frac{(m m' + m'^2) v'^2}{m+m'}$$

$$= \tfrac{1}{2} m v^2 + \tfrac{1}{2} m' v'^2$$

$=$ **total kinetic energy of system** ....(56)

It must here be observed that the *square* of a directed quantity (such as a velocity) is a quantity without direction ; it will always be *positive*, no matter what signs may be attributed to arbitrary directions in space. Thus all the expressions for kinetic energy involve the square of a velocity, and two or more quantities of energy may therefore be added together in the same way as two masses or two pure numbers. For this reason the ordinary signs + and − were used in place of $\ddagger$ and $\leftrightarrow$ in the last three lines of the above. The two terms

involving $v v'$ destroy one another, and we need not therefore consider the nature of such a product.

265. From equation (56) we learn that the kinetic energy of two moving bodies may be divided into two parts ; one due to the movement of the bodies relatively to one another, or relatively to their centre of mass ; the other due to the translation of the mass-centre (relatively, of course, to some other body which furnishes a standard of position). More generally, let a system be considered which includes any portion of matter in the universe, however distributed, and imagine this system to be arbitrarily divided into any two parts A and B, whose masses are respectively M and N, and whose mass-centres have the relative velocity V. Then the internal kinetic energy of the whole system may be divided into two parts: one part,

$$\tfrac{1}{2}\,\mathrm{M\,N\,V^2/(M + N)},$$

due to the motion of A relatively to B, and the other part,

$$\tfrac{1}{2}\,(m_1\,u_1{}^2 + m_2\,u_2{}^2 + \ldots) + \tfrac{1}{2}\,(n_1\,v_1{}^2 + n_2\,v_2{}^2 + \ldots),$$

due to motions internal to A and to B ; m_1, m_2, &c., being the various moving masses which constitute the portion A, and u_1, u_2, &c., their velocities relative to A's centre of mass, while n_1, n_2, &c., and v_1, v_2, &c., are the corresponding quantities for the portion B. If the system considered is our planet, and if the portion A is a body of small mass at its surface, M will be extremely small compared with N, and the fraction N/(M + N) will be almost exactly unity, so that the expression for the relative kinetic energy in this case reduces to $\tfrac{1}{2}\mathrm{M\,V^2}$, the form which is generally assumed in practice. Since the earth, however, is rotating on its axis, different parts of its surface have different velocities relatively to its centre of mass, and terrestrial motions are really measured with reference to the nearest parts of the surface; but the same principle will also apply here, for every body with which we can experiment is so insignificant compared with the earth that, if v be its velocity relatively to the nearest part of the surface, its kinetic energy may be taken as $\tfrac{1}{2}\,m\,v^2$ *this being the amount of work*

which will be performed by the body when brought to rest relatively to the surface.

266. To estimate completely the internal kinetic energy of a material system is not easy; for after account has been taken of the motion of all appreciable parts, there still remains the motion of the molecules relatively to one another, and the internal motion of the individual molecules. In accordance with the principles already laid down, the energy thus arising must be added to the relative kinetic energy of appreciable aggregations of molecules, in order to find the internal kinetic energy of the system.

267. **The Potential Energy** of a system cannot be absolutely defined. For example, when a mass m descends through a height h, the potential energy between that mass and the earth is diminished by $m g h$, for so much capacity for work has been expended; but we cannot well decide when the stone has no more potential energy, for after it has reached the ground it may still be made to descend farther by taking it to lower ground, or by dropping it down a mine; and when a material system consists of many separate bodies, each in turn made up of countless atoms acting and reacting on one another with forces of various kinds, it is impossible to say what configuration corresponds to the irreducible minimum of potential energy. All that we require to know, however, is the *gain or loss* of potential energy when the system changes from one configuration to another.

268. It will now be necessary to examine more closely what is meant by potential energy, and to this end let the following example be considered (fig. 138). A B is an inclined plane, and a body of mass m is moved along the plane from A to B. Draw A C vertical and B C horizontal, and let A B $= l$, B C $= b$, A C $= h$. Then the weight of the mass m is a force $m g$ acting

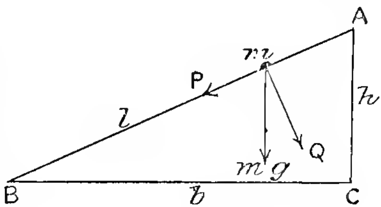


Fig. 138.

vertically downward, and A C ($= h$) is the component of the displacement A B resolved in the direction of the force, so that the work done by gravity upon the body during its descent is $m g h$, the same as if it had fallen vertically from A to C. The same conclusion may also be established as follows: let the force $m g$ be resolved into two components, one, $Q = m g \cdot b/l$, perpendicular to the plane, the other, $m g \cdot h/l$, along the plane. Then the displacement A B is perpendicular to the former component, which therefore does no work, positive or negative; the latter component, $m g \cdot h/l$, acts through a displacement $A B = l$ in its own direction, and performs, therefore, an amount of work $m g \cdot (h/l) \cdot l = m g h$, as before.

No account has here been taken of the possible roughness of the plane, or of any extraneous resistance which the mass may encounter in its descent; all that we wished to consider was the work done *by gravity* during the displacement A B. If there are forces such as friction opposed to the displacement, part of the work will be done against such forces, the remainder being expended in giving kinetic energy to the mass m. It is clear, then, that when a mass m passes by a rectilinear path from a point A to a point B, the work done *by gravity* is always $m g h$, where h is the vertical height of A above B; if B is on a higher level than A, h is of course negative, and so is $m g h$. The principle may also be extended; for a curved path of any form between A and B may be divided into a very large number of portions, each of which is approximately straight, and if h_1, h_2, &c., be the components of vertical descent corresponding to these infinitesimal paths, the vertical height H of A above B is the algebraic sum $h_1 + h_2 + \ldots$, and the work done *by gravity* during these displacements

$$\begin{aligned} &= m g h_1 + m g h_2 + \ldots \\ &= m g \mathrm{H}, \end{aligned}$$

which is independent of the form of the path between A and B. Similarly, if the body return from B to A by a path of any form, the work done *by gravity* will be $- m g \mathrm{H}$; so that *on the whole no work is done either by or against gravity when a*

body describes a closed curve ; a path, that is, which begins and ends at the same point.

269. The same principle may also be established from other considerations ; for if on the whole positive work were done by gravity during the description of such a path, an indefinite amount of work could be obtained by causing the journey to be repeated again and again ; while if on the whole positive work were done against gravity, the same result would follow by making the body travel round in the contrary direction ; and such an indefinite gain of work from gravitative attraction would be entirely contrary to experience.

270. **Conservative Forces.**—Gravity is an example of what are called *conservative* forces ; forces, that is, which depend only on the relative positions of the bodies between which they act, and not on the direction or magnitude of the motion of those bodies. The surface friction of solids, and the internal friction of liquids and gases are examples of *non-conservative* forces ; they depend on the direction of relative motion. for they always act in opposition to the motion.

The term *potential energy* may now be defined as follows : *the total work done by the conservative forces of a system while it changes from one configuration to another is equal to the loss of potential energy , or the gain of potential energy on changing from one configuration to another is equal to the total work which must be done meanwhile against conservative forces.*

271. **Conservative System.**—If all the forces of a system are conservative, and if a given change of configuration produces a diminution of potential energy W, this amount of work will have been expended in increasing the kinetic energy of the system, and hence we have the proposition that *in a conservative system which does not exchange energy with any other system, the sum of the kinetic and potential energies is constant.*

272. **Joule's Equivalent.**—If the system is not conservative —if, for example, there are frictional resistances to be overcome—a change of configuration may involve the expenditure of work against non-conservative forces ; but the energy so expended isnever destroyed, it is merely converted into some other form. Thus, energy expended against friction appears

mostly as heat, and heat is a form of energy which, like every other form, can be measured in ergs or in foot-poundals. The measurement may be effected by various methods, the most direct of which is due to Joule. A spindle S and a paddle P are rigidly connected together, and can turn about a vertical axis (fig. 139). The revolving paddle is surrounded by a

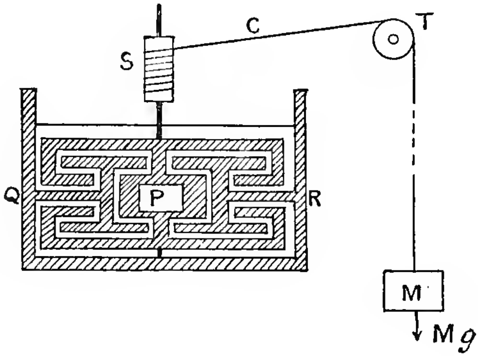


FIG. 139.

trough which contains water or mercury, and is provided with fixed paddles, Q and R. A thread C is wound many times round the spindle S, and passing over the smooth fixed pulley T, supports the mass M, whose weight is Mg. This mass, being allowed to descend, will set the apparatus in motion, and will fall but slowly, owing to the friction of the liquid, which is churned between the fixed and revolving paddles. Nearly all the work done by gravity during the descent has thus been expended against frictional resistance, and the liquid in the trough is found to have become warmer, its rise of temperature, as well as its mass, being measured. By such means it is found that about 42,000,000 ergs of work will produce sufficient heat to raise one gram of water from 0° to 1° Centigrade ; so that any quantity of water falling vertically through 42,800 metres in the latitude of London would acquire sufficient energy to raise its own temperature by one Centigrade degree. The many precautions

and corrections required in practice are necessarily left unnoticed in this brief description, our present object being merely to indicate how energy apparently lost may be traced in some other form.

Heat-energy is partly kinetic and partly potential; the kinetic energy is due to the relative motion of molecules and of the parts of molecules, the potential energy is due to the separation of mutually attracting molecules and atoms. When a body rises in temperature, the molecular motions become more rapid, and in most cases the body expands, increasing the degree of separation of its molecules, so that both the kinetic and potential energies are increased, and a corresponding amount of energy resides in the body in the form of heat.

273. **Conservation of Energy.**—The principle that energy, however transformed, is never destroyed, is called the *Conservation of Energy*, and is thus enunciated by Clerk Maxwell[1]: *The total energy of any body or system of bodies is a quantity which can neither be increased nor diminished by any mutual action of these bodies, though it may be transformed into any of the forms of which energy is susceptible.*

It may be remarked that every material phenomenon involves transformations of energy, and that the principle just enunciated is of constant application in all branches of physical science, for it sometimes enables us to predict unobserved phenomena, and to estimate unmeasured quantities, while it supplies a most important check upon theoretical speculations; and it is one of the strongest evidences of the universal truth of the principle that when logically applied it has never been known to lead to erroneous conclusions.

274. **Energy Transformed by Impact.**—As an example of the transformation of energy, consider the direct impact of two bodies whose masses are m and m', their velocities before impact being u and u'. We know from §§ 264, 265 that the total kinetic energy may be divided into two parts; one due to the motion of the mass-centre of the system, the other, $\frac{1}{2}\, m\, m'\, (u - u')^2/(m + m')$, due to the relative motion of the spheres. The velocity of the mass-centre being unaffected by the

[1] *Theory of Heat*, 4th ed., pp. 92, 93.

impact, the former portion of the energy does not change; but since the relative velocity after impact becomes $-e(u-u')$ the second portion of the energy is reduced to

$$\frac{m\,m'\{-e(u-u')\}^2}{2(m+m')} = e^2 \cdot \frac{(u-u')^2}{2(m+m')},$$

where e is the coefficient of restitution.

Thus the internal kinetic energy of the system is changed in the ratio of $e^2 : 1$, and since e is never greater than unity, there is always a diminution of kinetic energy, except in the case when e is equal to unity and the energy remains unchanged. Whatever kinetic energy may have disappeared from the system has been chiefly transformed into heat, some small portion, however, being communicated to the air in the form of sound-vibrations, while some, perhaps, may have been expended in producing electrification, and so forth.

275. **Applications.**—The principle of the conservation of energy may often be profitably applied in cases where no work is expended in friction or in imperfectly elastic concussions; that is, where the sum of the kinetic and potential energies of appreciable masses is constant. For example, if a pendulum A B (fig. 140) be supported on a frictionless knife-edge at C, it will rest with G, its centre of gravity, vertically below C; and if it be displaced so that the centre of gravity is at G′, and then be allowed to swing, the centre of gravity will rise to a point G″ on the same level as G′. For when the centre of gravity was displaced to G′, the pendulum had gained potential energy equal its weight multiplied by the vertical height of G′ above G; this being the amount of work which must be done against gravity to produce the displacement in question. As the pendulum returns towards its equilibrium position, the potential energy diminishes, and since no work is

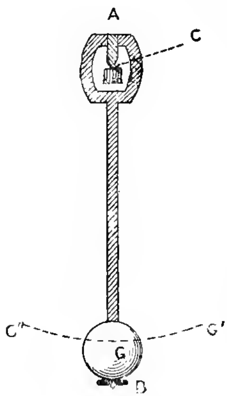


Fig. 140.

spent in overcoming resistances, the kinetic energy must increase at exactly the same rate. When the centre of gravity has reached G, the lowest point of its path, all the potential energy of the original displacement has been transformed into kinetic energy, and then the reverse process begins, and continues until the pendulum comes to momentary rest at the other limit of its swing, when all the energy is once more potential. In practice, the friction of the knife-edge and the resistance of the air will prevent these conditions from being exactly realised, but it is possible so to reduce the disturbing influences that the above remarks are very near the truth.

276. A very important principle is the following : *the conservative forces of a system always tend to produce displacements in such directions that the potential energy is diminished.* For suppose that the configuration A possesses more potential energy than the very slightly different configuration B ; then positive work must be expended against the conservative forces to bring the system from the configuration B to the configuration A, that is, the conservative forces will *oppose* the displacement from B to A, and will therefore favour the displacement from A to B, since by definition (§ 255) they are determined wholly by the configuration, and act independently of the direction of motion.

277. The theorem of § 202 may be established from these considerations, the only conservative force being the weight of the body. Since the constraints are smooth and rigid, the resistance which they exert at any point of the body is always perpendicular to its displacement, and no work is therefore expended on the constraints. We have to determine then what displacements are permitted by the constraints, and to determine for which of these (if any) the potential energy begins to decrease, for which to increase, and for which it remains stationary ; and since the potential energy here depends on the height of the centre of gravity, we arrive at the same conditions of equilibrium as were previously enunciated. Moreover, if a small displacement has raised the centre of gravity, positive work has been done against gravity to produce the displacement, and the weight acts, therefore, so as to oppose it. Thus the body tends to return to its former position, and the equilibrium

is stable for the given small displacement ; the conditions for unstable and neutral equilibrium being similarly deduced.

278. **The Principle of Virtual Work** may be enunciated as follows : *If a system of forces is in equilibrium, and if the points of application of the forces receive small displacements, the total work done by the forces is zero.* It is of course supposed that the displacements are consistent with the constraints of the material system on which the forces act, and it is further supposed that the displacements are so small that the forces do not appreciably change. A simple case is that in which the forces act on a rigid body, for if the body receive a small displacement, the total work done by the forces will be equal to the product of their resultant by the component of the displacement resolved in the direction of the resultant, and since the resultant is zero, the work done is also zero. More generally it is evident that when the forces are in equilibrium throughout the displacement, no portion of the material system experiences, on the whole, any force ; the work done by any given force during the displacement being expended against those forces which balance it, and *vice versâ*, so that the entire work done by the forces is zero—*observe that this need no longer be the case if the displacements are such that the forces meanwhile change appreciably, for equilibrium may not then obtain.*

279. **Applications.**—The principle of virtual work is easily applied to calculate the mechanical advantage of simple machines. Supposing that the working parts are frictionless, and that no positive or negative work is done in raising or lowering heavy parts of the machine, the work expended by the power will be done entirely against the weight; so that if p is the displacement of P, measured in the direction of P, and w that of W measured in the direction of W, Pp will be the work done *by* the power, and Ww the work done *by* the weight, and we shall have

$$\mathrm{P}p + \mathrm{W}w = 0, \qquad \qquad (57)$$

the mechanical advantage W/P being equal to $-p/w$. The following examples will serve to illustrate the application of the principle :

(1) *The Lever.*—Let the lever A B have its fulcrum at C (fig. 141), and let it receive a small displacement, which brings it to the position A′ B′. Let P and W act vertically downwards at B and A respectively so as to balance one another, and draw

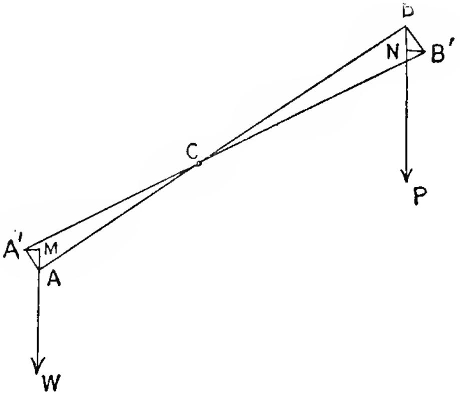


FIG. 141.

B′ N, A′ M horizontal, as in the figure. Then B N is the component of the displacement of P, resolved in the direction of P, while A M is the displacement of W resolved in the direction of W, and since A M and W are in contrary directions, the product W. A M will be negative. The equation of virtual work

$$P \cdot BN + W \cdot AM = 0$$

gives for the mechanical advantage

$$W/P = -BN/AM = BC/CA,$$

a result previously obtained.

(2) *The First System of Pulleys* (fig. 142).—Let the power P receive a displacement p in its own direction; the pulley A evidently rises by an amount equal to $\frac{1}{2} \cdot p$, causing B to rise through $(\frac{1}{2})^2 \cdot p$, C through $(\frac{1}{2})^3 \cdot p$, and, finally, D (which supports the 'weight') through $(\frac{1}{2})^4 \cdot p$, this displacement being oppositely directed to W. Thus, $-p/w = 2^4$, which gives, by the

principle of virtual work, $W/P = 2^4$, the movable pulleys being four in number.

(3) *The Inclined Plane, the 'Power' acting Parallel to the Plane.*—It is evident from the figure (143), if P receive a displacement p in its own direction, that w, the corresponding *vertical* displacement of W, is upward, so that Ww is negative; and we have further, from similar triangles, that p/w is equal in magnitude to l/h; that is, $W/P = -p/w$.

280. The same principle may easily be applied to the other simple machines,

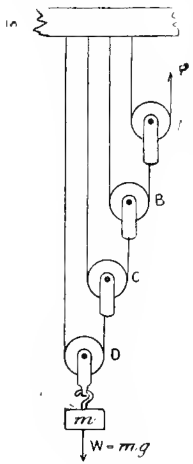


FIG. 142.

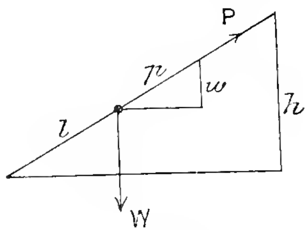


FIG. 143.

and in some cases it furnishes the most convenient method of investigation, as in the following examples:—

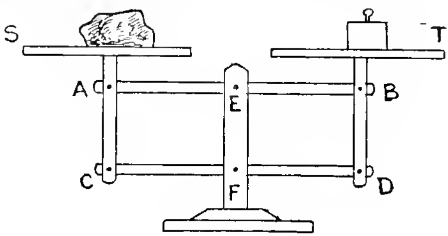


FIG. 144.

(1) *Roberval's Balance.*—This instrument is convenient for rough-and-ready measurements of mass, and is largely used by greengrocers and other tradesmen; it is constructed as

follows (fig. 144): A B, C D are rigid beams, joined by pivots with the rigid beams A C, B D, and capable of turning in a vertical plane about the fixed pivots E and F; while C A, D B (produced) support the scale-pans S, T. A E is made equal in length to C F, and E B to F D, A C and B D being each equal to E F; so that each of the figures A C F E, E F D B always remains a parallelogram; and E being vertically above F, A C and B D are always vertical. Thus when the scale-pan S is depressed, every point in it receives the same displacement, and the same is equally true of the scale-pan T. Consider, then, the points A and B: their displacements are always in opposite directions, and their magnitudes are as A E : E B; hence also,

$$\frac{\text{vertical fall of S}}{\text{vertical rise of T}} = \frac{\text{A E}}{\text{E B}}.$$

Suppose now that the instrument is in equilibrium when the scale-pans are empty, and the beams A B, C D horizontal; it will follow that equilibrium is not disturbed by placing in S and T masses proportional to E B and A E respectively, for the weights of these masses being in the same proportion, we have during any displacement,

$$\frac{\text{work done by weight of body in S}}{\text{work done against weight of body in T}} = \frac{\text{E B} \cdot \text{A E}}{\text{A E} \cdot \text{E B}} = 1.$$

Hence the weights of the two masses will be in equilibrium. If A E = E B, it follows that, when equal masses are placed in S and T, their weights will balance one another.

(2) *The Differential Screw* (fig. 145).—A beam K M of a fixed framework K L N M forms the nut of a right-handed screw A B. This screw itself is hollow, and forms the nut of another right-handed screw C D, which can only move up and down, its rotation being prevented by a bar E F, whose ends fit into smooth vertical grooves in K L, M N. Now let the screw A B receive one complete rotation in the direction of clock-hands, as seen from above; this will cause it to descend through a distance equal to its own pitch (§ 167), and at the same time A B will have descended, *relatively to* C D, through

a distance equal to the pitch of C D—that is, C D will have *risen* through this distance relatively to A B. The displacement of C D relatively to the frame is made up of its displacement relatively to A B, and A B's displacement relatively to the frame—that is, the pitch of C D *upwards* and the pitch of A B *downwards*, or the total downward displacement of C D = pitch of A B − pitch of C D.

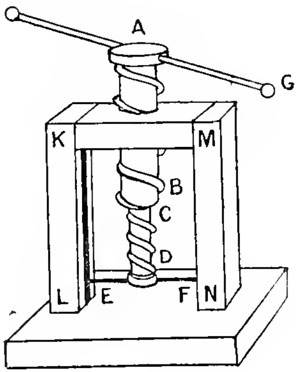


FIG. 145.

Now suppose that the 'power' P is applied at G horizontally and perpendicularly to A G, while the 'weight,' or force to be overcome, acts vertically upwards on C D. When equilibrium obtains between P and W, we shall have, by the principle of virtual work, so far as magnitude is concerned,

$$\frac{W}{P} = \frac{\text{displacement of P}}{\text{displacement of W}} = \frac{2\pi \cdot AG}{\text{pitch of AB} - \text{pitch of CD}};$$

and by making A B and C D very nearly equal in pitch, the mechanical advantage may be made as great as we please.

281. **Power.**—The term 'power' has already been used as a name for one of the forces applied to a machine; we have now to define its use in a sense so different that any confusion between the two meanings seems fortunately out of the question. *The power of an agent is the rate at which it can do work, and is measured by the number of units of work performed in the unit of time*; in the C.G.S. system the unit of power is *one erg per second*; in the British absolute system, the unit is *one foot-poundal per second*. In this country the unit chiefly employed by engineers is called a horse-power, and is equal to 33,000 foot-pounds per minute—that is, in one minute an agent of one horse-power could raise 33,000 pounds of matter through a height of one foot. The amount of work required for the

operation depends, of course, on the value of g; so that the horse-power, like all other gravitation units, has different values in different localities.

If a force P act on a body in a given direction, and if the body continues moving with velocity v in the same direction, the product Pv will measure the rate at which the force does work upon the body, for, in the unit of time, v will measure the space described, and Pv will be the work done.

Examples on Chapter XVI

(1) A body whose mass is 500 kilograms lies on a rough horizontal plane, the coefficient of friction being $\frac{1}{3}$; how much work must be expended to move the mass 30 metres along the plane by means of a horizontal force?

(2) A heavy body slides from rest down a rough plane, which is inclined 30° to the vertical, and whose length is l. If the velocity thus acquired is v, find how much work has been done against friction, and deduce the value of the coefficient of friction.

(3) Two masses, of 8 and 15 grams respectively, are moving in directions at right angles to one another, the first with a velocity of 400, the second with a velocity of 300 centimetres per second. Find the energy of their relative motion.

(4) In example (3), find the total kinetic energy of the two bodies, and hence deduce the velocity of their centre of mass.

(5) If a body constrained in any manner is initially at rest, and is allowed to move under the action of its own weight, show that its centre of gravity can never rise to a point higher than its original position.

(6) Two spheres of lead, of equal mass, impinge directly, the one having a velocity of 3 metres per second, and the other of 5 metres per second in the opposite direction. By what amount will the temperature of the spheres be raised by the impact, assuming all the kinetic energy which disappears to have been transformed into heat? (The coefficient of restitution for lead spheres is $\frac{1}{5}$, and to raise the temperature of one gram of lead through one degree Centigrade, 1,300,000 ergs of heat are required.)

(7) A bead slides on a smooth wire of any form; when the velocity at one point of the path is given, show how to find the velocity at any other point.

(8) A tank, 10 metres long and 8 metres broad, is filled with water to a height of 3 metres, and the bottom of the tank is 2 metres above the surface of a river. Remembering that a cubic centimetre of water has a mass of one gram, and taking 980 (C.G.S.) as the acceleration of falling bodies, find how much potential energy is lost when the tank is emptied into the river.

(9) A given force acts on a body which is initially at rest; show that the energy acquired by the body in a given time is inversely proportional to its mass.

(10) Show that, when a number of bodies are raised through various heights, the whole work performed is the same as if each body had received a displacement equal to that of the centre of gravity of the system.

(11) An engine is used to pump water through a hose. If the water issues from the nozzle with a velocity v, show that kinetic energy is imparted to the water at a rate proportional to v^3.

(12) If a diagram be constructed whose abscissæ represent lengths of path described by a body, and whose ordinates represent the corresponding components of force acting on the body in the direction of its path, how will the work done be represented?

(13) Show, by means of such a diagram, that, when a body displaced from its equilibrium position experiences a restoring force proportional to the displacement, the work done against restoring forces is proportional to the square of the displacement.

(14) Find the horse-power required to propel a train of 100 tons' mass with a uniform velocity of 30 miles an hour, the resistance to motion being equal to $\frac{1}{30}$th of the weight of the train.

(15) In the third system of pulleys, find the displacement of each pulley when the 'power' is displaced through the unit of length, and hence deduce the equation (26*a*) of § 154.

(16) A drum of radius a is fixed on the middle of a spindle, whose radius is b. A string is attached at one end to a fixed beam, while its other end is wound round the right hand half of the spindle. A second string is attached by one end to the same beam, and its other end is similarly wound round the left-hand half of the spindle. These two strings, hanging vertically from the beam, serve to support the spindle and drum, while a third string has one end wound round the drum, and has a body hanging freely at the other end. Apply the principle of virtual work to find the ratio between the weight of this body and that of the spindle and drum when there is equilibrium.

(17) Apply the principle of virtual work to determine to what order of levers the oar belongs.

(18) A carriage stands on a level road; compare the force which must be applied to the shafts to move the carriage with that which would be necessary if applied horizontally at the highest point of a wheel.

(19) If the unit of force is 100 dynes, and the unit of velocity 5 metres per second, how many ergs per second are there in the unit of power?

(20) Five grams being the unit of mass, a metre the unit of length, and a minute the unit of time, determine the absolute unit of work.

CHAPTER XVII

ROTATION

282. **Uniform Motion in a Circle.**—Let a point move with a velocity of constant magnitude v round the circumference of a circle whose centre is at O (fig. 146). The velocity when at any point A of the circumference is perpendicular in direction to the radius O A, and equal in magnitude to v; it may therefore be represented, according to a certain scale, by the radius O K, perpendicular to O A. After a very short interval of time t, the moving point will occupy a position B on the circumference very near to A, and its velocity when at B may accordingly be represented by the radius O L, perpendicular to O B, the angle K O L being equal to the angle A O B, and K L being equal to A B. Now the velocity (=) O L may be formed by compounding the velocity (=) O K with the velocity (=) K L, and since during the short interval of time t the velocity has changed from (=) O K to (=) O L, K L represents in direction and magnitude the velocity w added during this interval. But since A and B are very neighbouring points, the arc A B is very nearly equal to the chord A B, that is, equal to K L, which represents w; and the chord A B, being described in the time t with uniform velocity v, is equal to $v\,t$.

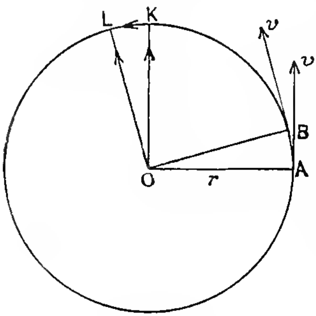


FIG. 146.

Again, since O K, K L, O L all represent velocities according to the same scale, we have

$$\frac{v}{\mathrm{OK}} = \frac{w}{\mathrm{KL}}$$

or, calling r the radius of the circle

$$\frac{v}{r} = \frac{w}{v\,t},$$

that is

$$\frac{w}{t} = \frac{v^2}{r};$$

the direction of w being the same as that of K L, that is, from the arc A B *towards the centre of the circle.* When the time t is taken sufficiently small, w/t, being the velocity added divided by the time, is the value of the acceleration, which is thus seen to be equal to v^2/r. *Hence the acceleration has the constant magnitude v^2/r, but varies continually in direction, being always directed towards the centre of the circle.*

283. If in the last section the moving point be replaced by a particle of mass m, the force required to produce the acceleration v^2/r is $m\,v^2/r$; and thus it follows that *if a body of mass m is moving with uniform speed v round the circumference of a circle whose radius is r, the resultant of all the forces acting upon the body is directed towards the centre of the circle, and is equal in magnitude to*

$$\frac{m\,v^2}{r} \quad \text{.............................} \quad (58)$$

284. **Angular Velocity.**—If a body is rotating uniformly about a fixed axis, its angular velocity is measured by the angle through which it turns in the unit of time, and the unit angular velocity is thus *one radian per second* (see § 30). From equation (9) we know that if r is the radius of a circle, $\theta\,r$ is the arc of its circumference which subtends an angle θ at the centre; and hence if a particle moving uniformly with speed v along the circumference has an angular velocity ω about the centre we have

$$v = \text{arc described in unit time} = \omega\,r;$$

and the acceleration of the particle towards the centre of the circle

$$= \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r,$$

the force urging the particle towards the centre of the circle being

$$m \omega^2 r \qquad \ldots\ldots (58\ a)$$

where m is the mass of the particle.

285. **Centrifugal Force and Centripetal Force.**—This force, which maintains the circular form of the path, is called *centripetal force* (*centrum*, the centre ; *peto*, I seek), and must be due to an action exerted between the whirling body and some other portion of matter, which in turn must experience an equal and opposite force, called *centrifugal force* (*fugio*, I flee). For example, neglecting gravity, take the case of a stone which is moving uniformly in a circle ; the force required for this motion may be supplied by a string which connects the stone to some fixed support at the centre of its circular path. The pull thus exerted on the stone by the central support is the centripetal force, and the equal and opposite pull exerted on the support by the stone is the centifugal force.

In the same way the earth moves round the sun in a nearly circular orbit, owing to the mutual attraction of these bodies ; or to speak more correctly, so far as the relative motion of the sun and the earth is concerned, each describes a circle about their common centre of mass ; but as this point is some hundreds of thousands of times farther from the earth's mass-centre than from the sun's, the proportion of the relative motion due to the latter body is comparatively insignificant.

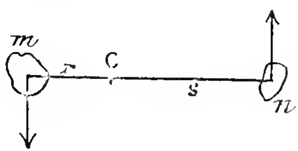


FIG. 147.

Let m and n (fig 147) be two masses which, owing to some mutual attraction, describe circular paths ; since the motion of their mass-centre (C) will not thus be affected, we shall consider the velocities of both masses relatively to this point.

Let the distances of C from the mass-centres of m and n be r and s respectively, and let ω be the angular velocity of the straight line $m\,n$, that is, the angular velocity with which either body revolves about C. Then the force exerted on the mass m in the direction m C must be equal to $m\,\omega^2\,r$, the force on n in the direction n C being similarly $n\,\omega^2\,s$; and since $m\,r$ is equal to $n\,s$, these forces are of equal magnitude, in conformity with the third law of motion.

It will thus be seen that the distinction between centrifugal and centripetal force is rather apparent than real, the meanings of the terms being interchanged according as we consider the motion of the one or of the other body.

286. **Rotation about a Fixed Axis.**—Let a particle of mass m be so constrained that it can only revolve about a fixed axis A B (fig. 148), the system being frictionless and the mass of the constraints being small enough to be neglected. We shall suppose either that gravity is absent or that the axis A B is vertical, so that the weight of the particle does not affect its motion, and if, under these circumstances, the system is set rotating with angular velocity ω, it will continue rotating uniformly, and the only force exerted upon the particle m will be $m\,\omega^2\,r$ directed towards the centre O of its circular path and due to the action of the constraints.

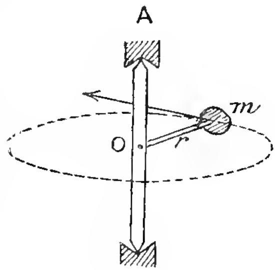


FIG. 148.

But suppose, now, that the system is acted upon by any forces whose moment about the fixed axis is M; these may be replaced by any other forces which have the same moment M about the fixed axis; for example, by a force M/r acting upon m in a direction tangential to its circular path. This force, then, will produce the acceleration $\dfrac{M}{r \cdot m}$ in its own direction, the velocity gained during a very short interval of time, t, being

$\frac{M}{r\,m}t$; and since $\omega = v/r$, the corresponding gain of angular velocity is

$$\frac{1}{r}\cdot\frac{M}{r\,.\,m}\cdot t, \text{ or } \frac{M}{m\,r^2}\cdot t;$$

whence $\frac{M}{m\,r^2}$ is the *angular acceleration* or *rate of change of angular velocity*, which may be denoted by ϕ, so that

$$\phi = \frac{M}{m\,r^2}, \text{ or } M = m\,r^2\,.\,\phi;$$

the moment M *required to produce an angular acceleration* ϕ *being* $m\,r^2\,.\,\phi$.

Now, suppose there are a number of particles m_1, m_2, . . at distances r_1, r_2, . . respectively from the same fixed axis (fig. 149), about which each can rotate independently and without friction. If no forces act upon the particles except the constraint of the fixed axis, their angular velocities ω_1, ω_2, . . . will all remain unchanged, and if, further, $\omega_1 = \omega_2 = \ldots$ the system of particles will rotate round the axis as if they formed part of a single rigid body. Let forces now act on the particle m_1

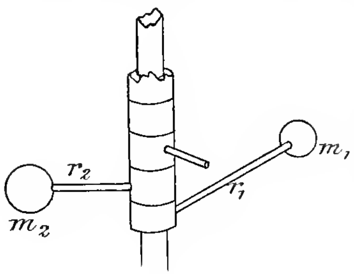


FIG. 149.

which have a moment $m_1\,r_1^{\,2}\,\phi$ about the fixed axis; they will produce an angular acceleration ϕ in the motion of the particle, and if at the same time forces act on the remaining particles whose moments about the fixed axis are $m_2\,r_2^{\,2}\,\phi$ . . . respectively, each of these particles will have the same angular acceleration ϕ, and the system will still rotate like a single rigid body. The motion, then, will not be altered by supposing the particles to be rigidly connected together, and we have thus a rigid body rotating about a fixed axis and acted upon by forces whose moment about this axis is

$$m_1 r_1^2 \cdot \phi + m_2 r_2^2 \cdot \phi + \ldots,$$

or $$(m_1 r_1^2 + m_2 r_2^2 + \ldots) \phi = \text{M (suppose)}.$$

These forces may (by § 158) be replaced by any others which have the same moment about the fixed axis, and we thus arrive at the following result: *If a rigid body capable of rotating about a fixed axis be considered as made up of particles whose masses are* m_1, m_2, . . . *and whose distances from the axis are respectively* r_1, r_2, . . . , *and if the body be acted upon by forces whose moment about the axis is* M, *the angular acceleration produced will be*

$$\phi = \frac{\text{M}}{m_1 r_1^2 + m_2 r_2^2 + \ldots} \quad \ldots\ldots\ldots\ldots(59)$$

287. **Moment of Inertia.**—In the case of an unconstrained body we have seen that

$$f = \frac{p}{m},$$

where m is the mass of the body, p the resultant of all the forces acting upon it, and f the acceleration of its mass-centre.

This equation is identical *in form* with (59), the *acceleration* f corresponds to the *angular acceleration* ϕ, the *force* p to the *moment* M, and the *mass* or *inertia* m to the quantity $m_1 r_1^2 + m_2 r_2^2 + \ldots$, which is called the *moment of inertia* of the body about the fixed axis.

The moment of inertia, then, has the same relation to rotation that mass has to translation:

For translation: **force = mass × acceleration.**

For rotation. **moment = moment of inertia × angular acceleration.**

Let K denote the moment of inertia of a given body about a given axis; then, to find K, we must suppose the whole substance of the body to be divided up into very small portions, and the mass of each portion to be multiplied by the square of its distance from the given axis. K will then be equal to the sum of all the products so formed. By dividing the body into sufficiently small parts, the moment of inertia may be found as accurately as we please.

288. **Radius of Gyration.**—Let k be such a length that

$$m\,k^2 = m_1\,r_1{}^2 + m_2\,r_2{}^2 + \ldots = \mathrm{K},$$

$m\,(= m_1 + m_2 + \ldots)$ being the mass of a given body and K its moment of inertia about a given axis; then k is called the *radius of gyration* about this axis, the moment of inertia being evidently the same as if every particle were at a distance k from the axis of rotation.

289. **Kinetic Energy of Rotation.**—When a body is rotating with angular velocity ω, the velocity of a particle distant r_1 from the axis of rotation is $\omega\,r_1$, and the energy due to the motion of this particle is $\frac{1}{2}\,m_1 \cdot \omega^2\,r_1{}^2$; m_1 being its mass. The whole kinetic energy of rotation is thus

$$\begin{aligned} &\tfrac{1}{2}\,m_1 \cdot \omega^2\,r_1{}^2 + \tfrac{1}{2}\,m_2 \cdot \omega^2\,r_2{}^2 + \ldots \\ &= \tfrac{1}{2}\,(m_1\,r_1{}^2 + m_2\,r_2{}^2 + \ldots)\,\omega^2 \\ &= \tfrac{1}{2}\,\mathrm{K}\,\omega^2, \end{aligned}$$

which is the analogue of the expression $\frac{1}{2}\,m\,v^2$ for the energy of a translational movement.

It has already been shown (§ 264) that the kinetic energy of a material system may be divided into two parts, one due to the motion of translation of the mass-centre (relatively to other matter), the other due to the motion of the component masses relatively to the mass-centre. Now if the system consists of a single rigid body, the only relative motion possible is that involved in the rotation of the body, which causes the distances of particles from the mass-centre to alter in *direction*, though not in magnitude. Thus the complete expression for the kinetic energy of a rigid body is

$$\tfrac{1}{2}\,m\,v^2 + \tfrac{1}{2}\,\mathrm{K}\,\omega^2,$$

where m is its mass, v the velocity about its mass-centre, K the moment of inertia about the axis of rotation through the mass-centre, and ω the angular velocity.

290. It should be noticed that although translational movements are essentially relative, the same is not true of rotations; in fact, as we have just seen, the *rotation* of a body involves the *relative motion* of its parts. If a body could be completely

isolated from all other matter, we should have no means of determining the velocity of its mass-centre, nor indeed could any physical meaning be attached to the term; but if rotation takes place, centrifugal and centripetal forces will be exerted between the various parts of the body, and from a knowledge of these forces the rotation could be determined.

Examples on Chapter XVII.

(1) A mass of 250 grams is attached to one end of a string 1·5 metres in length whose other end is fixed, and the mass describes uniformly a horizontal circle while the string is always inclined 30° to the vertical. Find the pull of the string, and hence deduce the velocity of the moving mass, assuming the acceleration of falling bodies to be 980 (C.G.S.).

(The forces acting on the body are its weight and the pull of the string, and these must have a horizontal resultant directed towards the centre of the body's circular path. Thus the magnitude of this resultant and of the pull of the string can be found, and accordingly the speed of the body along its circular path can be deduced.)

(2) If at a given instant a body is rotating about a fixed axis with angular velocity ω, and is subject to a constant angular acceleration ϕ about the same axis, show that the angle θ described in the time t from the given instant is determined by

$$\theta = \omega t + \tfrac{1}{2}\phi t^2.$$

This equation is analogous to equation (6), § 23.

(3) A body is constrained to rotate without friction about a fixed axis, and is subject to the action of forces whose moment about the fixed axis is 48,000 dyne-cm. If the angular acceleration produced is 400 radians per second per second, and the mass of the body 60 grams, find the radius of gyration about the fixed axis.

(4) If a body capable of rotation about a fixed axis consists of several parts whose masses are m_1, m_2, &c., and whose radii of gyration about the fixed axis are k_1, k_2, &c., show that the radius of gyration of the whole body about the same axis is

$$\sqrt{\left(\frac{m_1 k_1^2 + m_2 k_2^2 + \ldots}{m_1 + m_2 + \ldots}\right)}$$

(5) Show that a body rotating about a fixed axis has the same amount of kinetic energy as if its mass were collected in a single particle whose distance from the fixed axis was equal to the radius of gyration.

(6) A uniform cylinder, whose mass is 50 kilograms and whose radius of gyration about its geometrical axis is $1/\sqrt{2}$ of its geometrical radius, rolls along the ground with a velocity of 4 metres per second. Find its kinetic energy.

CHAPTER XVIII

DIMENSIONS OF UNITS

291. In the foregoing chapters we have dealt with many quantities essentially different in kind from one another, and each measured in terms of its own unit; and yet it has only been found necessary to fix the values of three fundamental units —those, namely, of length, of mass and of time—all other dynamical units being deduced from these by simple definitions, without the introduction of any new arbitrary quantity. It will be seen as we proceed how the term '*dimensions*' is used in this connection.

292. **Length, Area, and Volume.**—The unit of area always adopted is the area of the square whose side is the unit of length, and the area of any surface may be expressed in terms of such a unit. Whatever be the form of the surface, a rectangle may always be described having an equal area, which is measured by the product of its two sides. Thus, in every case, an area may be represented as *a length multiplied by a length*, and if we use the symbol [L] to stand *not for any particular length, but for some quantity of the nature of a length*, then

$$[L] \times [L], \text{ or } [L]^2$$

will stand for *some quantity of the nature of an area.* This is otherwise expressed by saying that *area is of two dimensions in length*, or that *the dimensions of area are* $[L]^2$.

Similarly, *volume* is of the dimensions $[L]^3$.

293. **Angle.**—An angle being measured by the *ratio* of the length of a circular arc to the length of its radius, has the dimensions

$$\frac{[L]}{[L]}, \text{ or } [L]^0;$$

or, in other words, it is without physical dimensions, the definition of the unit angle being independent of all concrete units. At the same time it must be observed that the angle between two lines is a *directed quantity* ; for, to make one line coincide in direction with the other, we must turn it through an angle *about some axis which has a definite direction.*

294. **Mass and Time.**—Let [M] stand for *some quantity of the nature of a mass*, [T] for *some quantity of the nature of an interval of time.*

295. A **Velocity** may always be measured by length of path described, divided by time taken ; so that its dimensions

$$\frac{[L]}{[T]}, \text{ or } [L][T]^{-1} ;$$

in other words, *velocity is of one dimension in length and of minus one dimension in time.*

296. **Acceleration,** being measured by velocity gained, divided by time taken, has the dimensions

$$\frac{[L][T]^{-1}}{[T]}, \text{ or } [L][T]^{-2} ;$$

that is, *acceleration is of one dimension in length and of minus two dimensions in time.*

297. **Force** is mass-acceleration ; it is equal to the mass acted upon, multiplied by the acceleration produced, and its dimensions are therefore,

$$[M][L][T]^{-2}.$$

298. **Work** is measured by the force acting multiplied by the distance through which it acts in its own direction, and has therefore the dimensions

$$[M][L][T]^{-2}.[L], \text{ or } [M][L]^2[T]^{-2}.$$

299. The **Kinetic Energy** of a body is equal to half the product of its mass by the square of its velocity ; hence its dimensions are

$$[M]\{[L][T]^{-1}\}^2, \text{ or } [M][L]^2[T]^{-2} ;$$

energy, as we have already seen, being a quantity of the same

kind as work. The factor $\frac{1}{2}$ of the expression $\frac{1}{2} m v^2$, being a purely numerical quantity, does not of course enter into the dimensions; it modifies the magnitude but not the nature of the quantity expressed.

300. The **Moment** of a force about a point is the product of the force and the arm on which it acts; it has the dimensions

$$[M][L][T]^{-2}.[L], \text{ or } [M][L]^2[T]^{-2},$$

which are the same as those of work or energy. A moment, then, should be measurable in terms of the erg, or of any other unit of work: Let the moment P.O A produce a rotation through the angle A O B, which is equal to one radian (fig. 150). Then the force P has moved its point of application through the arc A B, whose length is equal to O A, and the work done is therefore P.O A, which is the moment of P about O. Hence a moment is measured by the work which it does in producing a rotation of one radian, and since a radian is defined without reference to any concrete magnitude, it is evident that, in order to fix the unit moment, we have only first to fix the unit of work.

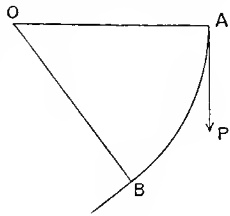


Fig. 150.

There is, however, another important feature which we have not included in the dimensions; that is, *direction*. Work is the product of a force and a displacement in its own direction, and is *not* a directed quantity; a moment is the product of a force and an arm perpendicular to its direction, and is itself a directed quantity, for it tends to produce rotation in the plane of the force and the arm, that is, about an axis perpendicular to this plane.

301. **Momentum** is the product of a mass and a velocity, and its dimensions are

$$[M][L][T]^{-1}.$$

302. The **Impulse** of a force is the product of the force and the time during which it acts, and has also the dimensions

$$[M][L][T]^{-2}.[T], \text{ or } [M][L][T]^{-1};$$

the impulse of a force during any interval of time being measurable as a change of momentum.

303. **Power**, being the rate of doing work, is equal to the work done divided by the time taken, and so its dimensions are

$$\frac{[M][L]^2[T]^{-2}}{[T]}, \text{or } [M][L]^2[T]^{-3}.$$

304. It is not essential that length, mass and time, should be chosen for the fundamental units ; *any three independent units* would be theoretically sufficient, and the actual selection is only a matter of convenience. The term 'independent' here means that two of the chosen units must not fix the value of the third ; for example, mass, velocity and momentum would not furnish three independent units, for if any two of these units were chosen, we could immediately deduce the third. As an instance of three independent units, those of length, force and time may be mentioned. The unit of force in this system is usually the weight of a standard body at the sea-level in a given latitude (that is, in some place where g has a given value g_0). The absolute unit of mass is *not* the mass of this same standard body, but *the mass in which the unit of force produces the unit of acceleration*, or g_0 times the mass of the standard body. Such a system as this is just as much 'absolute' as the C.G.S. system, but it is not nearly so convenient, for a permanent unit of mass is much more easily accessible than an equally permanent unit of force.

Before calculating the dimensions of any quantity, the nature of all three fundamental units must be decided. Thus, when length, mass and time are the fundamental units, *work* is of one dimension in mass, of two dimensions in length and of minus two in time, while in a length-force-time system, work is of one dimension in force, of one dimension in length and of no dimensions in time. There is not, however, any real disagreement between these results, they are two different expressions of the same fact. For if [*force*] be replaced by its dimensions in length, mass and time, the expressions $[L][M]^2[T]^{-2}$ in the one system, and [*force*] [*length*] in the other, are seen to be exactly equivalent.

305. **Theorem.**—*If a quantity* K *be of l dimensions in length, and if the unit of length be increased n-fold, the unit in terms of which* K *is measured will be increased in the proportion* n^l. For, let the dimensions of K be $[L]^l[M]^m[T]^t$; then λ, μ, τ being the units of length, mass and time, and κ the unit for quantities of the nature of K, we shall have

$$\kappa = \lambda^l . \mu^m . \tau^t . C,$$

where C is a quantity which may or may not be unity, but which is without physical dimensions, and does not therefore vary when the units λ, μ, τ are varied. Thus the new unit of length being $n\lambda$, the new unit of the nature of K will be

$$(n\lambda)^l \cdot \mu^m \cdot \tau^t \cdot C = n^l \cdot \kappa ;$$

and the proposition is evidently perfectly general.

306. The practical expression of any physical magnitude involves two factors; one of these is the unit in terms of which such magnitudes are measured, the other is a pure number (integral or fractional) which indicates how many times the unit is contained in the magnitude considered. Thus we may speak of two grams, or five centimetres, or twenty dynes; but it is usual in theoretical work to represent a concrete physical by a single symbol.

A given body has a certain definite mass, and this may conveniently be represented by a symbol such as m, which does not refer to any special unit of mass, but stands simply for the mass of the body in question, whether we measure it in grams, or in grains, or in ounces. Now let M be adopted as the unit of mass, and suppose that this unit is contained μ times in the mass m, so that

$$m = \mu . M.$$

Thus we might have, for example,

$$m = 12\cdot5 \text{ grams}.$$

If the unit mass be changed from M to M′, the numerical measure of m will acquire some new value μ', where

$$m = \mu' \cdot M'.$$

Since m here stands for the same physical quantity as before, $\mu' M'$ must be equal to μM, or

$$\frac{\mu'}{\mu} = \frac{M}{M'};$$

in other words: *If we change the unit in terms of which a given magnitude is measured, the numerical measure of this magnitude will be changed in the inverse proportion.*

Examples on Chapter XVIII.

(1) If the unit of length be doubled, the unit of mass trebled and the unit of time halved, what effect will thus be produced on the units of momentum, of energy and of acceleration?

(2) When the fundamental units are those of length, of momentum and of velocity, find the dimensions of mass, of time and of energy.

(3) What will be the unit of power when the unit of energy is five ergs, the unit of length twenty centimetres and the unit of velocity 100 centimetres per second?

(4) A body has p times the unit of momentum and q times the unit of kinetic energy. Find its mass and velocity in terms of the fundamental units, namely, those of length, mass and time.

(5) Show that when a body capable of rotating about a fixed axis is acted upon by forces whose moment about that axis is given, the work done by the forces after a given time from rest is inversely proportional to the moment of inertia of the body.

ADDITIONAL EXAMPLES

CHAPTER I

(1) Explain how velocity is measured in dynamics. If the unit of space were a yard, and the unit of time half a minute, how would the units of velocity and acceleration be altered?

(2) Express in feet per second a velocity of 75 miles per hour.

(3) How many centimetres per second are equivalent to 15 metres per minute?

(4) Express a velocity of 44 feet per second in miles per hour.

(5) Compare the velocities of two bodies, of which one moves uniformly over a quadrantal arc of a circle, while the other describes the chord of the same arc.

(6) The earth being about 92,800,000 miles from the sun, and the velocity of light being 186,000 miles per second, how many minutes does the sun's light take to reach the earth?

(7) If sound in air travels 1,090 feet per second, compare its velocity with that of light.

(8) Explain how acceleration is measured, and state what will be the unit of acceleration when a mile is the unit of length and an hour the unit of time.

(9) Express an acceleration of 1 metre per second per second in millimetres per minute per minute.

(10) If an acceleration be represented by f when the unit of time is a minute, how will it be represented when the unit of time is a second?

(11) Compare the accelerations of two bodies, one of which acquires uniformly a velocity v_1 in the time t_1 and the other a velocity v_2 in the time t_2.

(12) If the measure of an acceleration is 12 when a foot and a second are units, what will be its measure when the units are a yard and a minute?

(13) What velocity will be gained in one minute by a point whose acceleration is 25 feet per second per second? State your result in yards per minute.

(14) A body's velocity increases uniformly from 10 to 17 metres per second in the course of half an hour. Find its acceleration in centimetres per minute per minute.

CHAPTER II

(1) If a heavy body is thrown vertically up to a given height, and then falls back to the earth, show that, neglecting the resistance of the air, it passes each point of its path with the same velocity when rising and when falling.

(2) A ball is allowed to fall to the ground from a certain height, and at the same instant another ball is thrown upwards with just sufficient velocity to carry it to the height from which the first one falls. Show when and where the two balls will pass each other.

(3) Prove the formula which gives the distance described in a given time by a body thrown vertically upwards with a given velocity.

(4) If the velocity of a body increase from 12 to 13 feet per second while it moves over a distance of 5 feet, what is the acceleration?

(5) Through what vertical distance must a heavy body fall from rest in order to acquire a velocity of 161 feet per second? If it continue falling for another second, after having acquired the above velocity, through what distance will it fall in that time? (Gravity = 32·2.)

(6) A balloon has been ascending vertically at a uniform rate for 4·5 seconds, and a stone let fall from it reaches the ground in 7 seconds. Find the velocity of the balloon, and its height when the stone is let fall.

(7) A body falling with a uniformly accelerated motion, passes through 10 feet in the first two seconds after starting. How far will it be from the starting-point at the end of the third second?

(8) A heavy particle is dropped from a point, and after it has fallen for one second another particle is dropped from the same point. What is the distance between the two particles when the first has been moving during 5 seconds?

(9) Define uniform acceleration. A particle moves with a uniform acceleration f. If v be the velocity at any instant, show that the space described in the time t after that instant is $vt + \frac{1}{2}ft^2$.

(10) A stone, after falling for one second, strikes a pane of glass, in breaking through which it loses half its velocity. How far will it fall in the next second?

(11) How far will a body fall in four seconds? With what velocity must a body be thrown vertically upwards in order to return to the hand after four seconds?

(12) An arrow is shot vertically upwards with a velocity of 100 feet in a second when it leaves the bow. How long will it be before it reaches the ground again?

(13) A heavy body is let fall from rest *in vacuo*. How far does it fall in the first, second, and fifth seconds respectively?

(14) A body is projected vertically upwards with a velocity of 89 feet per second. In what time will it attain a height of 149 feet?

(15) A body starts from rest and moves with uniform acceleration 18 (feet, seconds). Find the time required by it to traverse the *first*, *second* and *third* foot respectively.

(16) What is meant by saying that the acceleration of a particle is

32 foot-second units? With this acceleration how far will the particle move in 10 seconds, and what will be its velocity at the end of that time?

(17) A train, which is uniformly accelerated, starts from rest, and at the end of 3 seconds, has a velocity with which it would travel through 1 mile in the next 5 minutes. Find the acceleration.

(18) A bullet is shot vertically upwards with a velocity of 1,600 feet per second. Find how high it will go and how long it will be in the air, neglecting the resistance of the air.

(19) A particle moves from rest with a uniformly increasing velocity. Show that the whole space described is proportional to the square of the time.

(20) A stone dropped into a well reaches the water with a velocity of 80 feet per second, and the sound of its striking the water is heard $2\frac{7}{12}$ seconds after it is let fall. Find from these data the velocity of sound in air ($g = 32$).

(21) A body is allowed to fall freely from rest. Find an expression for its velocity at any point in terms of the distance through which it has fallen, and the acceleration of gravity. If $g = 981$ centimetre-second units, from what height must a body fall in order that it may have a velocity of 50 metres per second on striking the ground?

(22) What is the average velocity of a body during the first second of its fall under gravity; also during the first two seconds? Show how, by knowing the average velocity, you can find the whole space fallen through.

(23) There are two chambers, one of which is in the act of dropping freely under gravity down a pit, while the other is made to descend with uniform velocity. A man in each chamber during the descent lets go a stone which he has been holding in his hand. What will be the motion of the stone in each case?

(24) A body projected vertically upwards against gravity has risen 120 feet in one second. What was its initial velocity of projection, and how far will it rise during the next second?

(25) A stone projected vertically upwards reached the ground again in 6 seconds. What was its height above the ground at the end of the first second?

(26) A rifle-bullet is shot vertically downwards from a balloon at the rate of 400 feet per second. How many feet will it pass through in 2 seconds, and what will be its velocity at the end of that time, neglecting the resistance of the air and estimating the acceleration due to gravity at 32?

(27) At the earth's equator the hot air ascends and is replaced by cold air which blows in along the ground from the poles. That which comes from our hemisphere blows from the north-east instead of from the north. Explain this.

(28) If a body is projected vertically upwards with a velocity of 120 feet in a second, what is the greatest height to which it will rise, and when will it be moving with a velocity of 40 feet per second?

(29) A stone is let fall from the top of a tower. A second later, another stone is thrown vertically downwards after it and overtakes the first stone in a second. With what velocity was the second stone projected?

(30) The speed of a railway train increases uniformly for the first 3 minutes after starting; and during this time it travels 1 mile. What speed (in miles per hour) has it now gained, and what space did it describe in the first 2 minutes?

CHAPTER IV

(1) Explain a convenient method of representing geometrically the velocity of a body moving according to a known law and the distance passed over by it.

(2) Employ the above method to find the distance traversed in 10 minutes by a train which has a velocity of 20 miles an hour and which has its speed diminished at a uniform rate to 5 miles an hour.

(3) Apply the same method to determine the space passed over by a body in 10 seconds after it starts from rest and has its velocity increased by 1 foot per second at the beginning of each second.

(4) The abscissæ of a diagram represent time on a scale of 1 centimetre to the minute, and the ordinates represent velocity on a scale of 1 centimetre to 1 metre per minute. Draw the velocity-curve for a body which begins moving at 2 centimetres per second and during 10 seconds uniformly changes its velocity to 5 centimetres per second in the opposite direction. Hence calculate the distance described by the body.

(5) The abscissæ of a diagram represent time on a scale of 1 inch to the second, and a straight line on the diagram sloping upwards at an inclination of 45° corresponds to a motion uniformly accelerated to the extent of 10 feet per second per second. On what scale do the ordinates represent velocities, and what area on the diagram corresponds to a path 15 feet in length?

(6) Find the (uniform) acceleration of a body whose velocity-line is inclined 30° to the 'time'-axis, the abscissæ representing time on a scale of 3 centimetres to the minute, and the ordinates velocity on a scale of 5 centimetres to 1 metre per minute.

(7) Show by means of a diagram that the distance described in any time by a body moving irregularly along a rectilinear path is the same as would be described in the same time by a body whose velocity was uniform and equal to the mean velocity of the actual body.

(8) The abscissæ of a diagram represent time on a scale of 1 centimetre to the second, and an area of 5 square centimetres represents a displacement of 7 metres. What velocity is represented by an ordinate 3 centimetres in length?

(9) The ordinates of a diagram represent acceleration on a scale of 1 centimetre to one metre per second per second, and an area of 1 square centimetre represents a velocity of 15 metres per second. On what scale do the abscissæ represent time?

(10) Explain by reference to a diagram why a stone falls only 16 feet during the first second, while yet the force of gravity generates in that time the velocity of 32 feet per second.

CHAPTER VI

(1) Two velocities are represented by A P and A Q, while their resultant is represented by A R. If A P be fixed while the extremity Q describes any given circle, prove that the extremity R will describe another circle.

(2) The directions of two velocities are inclined to each other, (*a*) at an angle of 60°, (*b*) at an angle of 120°; and the respective resultants are in the ratio $\sqrt{7} : \sqrt{3}$. Compare the magnitudes of the velocities.

(3) A boat is rowed on a river so that its speed in still water would be 6 miles an hour. If the river flows at the rate of 4 miles an hour, show, by drawing a figure, how to find the direction in which the head of the boat must be kept in order that its motion may be at right angles to the current.

(4) Explain the rules for the composition and resolution of velocities and accelerations. Determine the direction taken by the smoke of a steamer.

(5) A balloon is carried along at a height of 100 feet from the ground with a velocity of 40 miles an hour; a stone is dropped from it. Find the time before the stone reaches the ground, and the distance from the point where it reaches the ground to the point vertically below the point where it left the balloon.

(6) Two bodies start together from rest, and move in directions at right angles to each other. One moves uniformly with a velocity of 3 feet per second; the other moves with a constant acceleration. Determine this acceleration if the bodies at the end of 4 seconds are 20 feet apart.

(7) A river 1 mile broad is running downwards at the rate of 4 miles an hour, and a steamer moving at the rate of 8 miles an hour wishes to go straight across. How long will the steamer take to perform the journey, and in what direction must she be steered?

(8) A balloon is carried along by a current of air moving from east to west at the rate of 60 miles an hour, having no motion of its own through the air; and a feather is dropped from the balloon. What sort of path will it appear to describe, as seen by a man in the balloon?

(9) Suppose that at the equator a straight hollow tube were thrust vertically down towards the centre of the earth, and that a heavy body were dropped through the centre of such a tube. It would soon strike one side. Find which, giving a reason for your reply.

(10) When a carriage travels with velocity v, find the apparent velocity of any point in one of the wheels, as seen by a person in the carriage; hence show that the highest point of the wheel moves with velocity $2v$, and that the lowest point is at rest.

CHAPTER VII

(1) A heavy particle is dropped from a height of $178\frac{8}{9}$ feet above a level plain, and while falling it is carried horizontally with a uniform velocity of 3 feet per second. At what distance from its starting-point will it strike the ground? ($g = 32 \cdot 2$.)

(2) A body is projected horizontally from the top of a tower with a velocity of 50 feet per second. Determine approximately the distance from the base of the tower at which it will strike the ground; and show that it will take as long to reach the ground as if it had been let fall instead of being projected.

(3) Find the time in which a projectile reaches its greatest height, and the greatest height.

(4) A ball is projected in a horizontal direction from a rifle placed 1,000 feet above the level of the sea. Find the elevation of the ball above the level of the sea 2 seconds after the discharge, neglecting the resistance of the air.

(5) Two particles, P and Q, are projected at the same instant from the same point, one with velocity v at an angle of 30° to the horizon, the other with velocity $\sqrt{3}\, v$ at an angle of 60°. Will they ever hit each other?

(6) If a bullet be fired from the earth in any direction, show that it will strike the earth again with the same velocity as that with which it was projected (neglecting the action of the air). What will be the direction of the motion when it is at its greatest elevation?

(7) A cannon-ball is shot horizontally from the top of a tower 49 feet high, with a velocity of 200 feet per second. Find at what distance from the tower the cannon-ball will strike the ground.

(8) Find the vertical and horizontal components of the velocity which must be given to a ball that it may pass horizontally over the top of a wall 50 yards off and 75 feet high.

(9) A stone is projected into the air with a velocity of 200 feet per second, in a direction inclined at 60° to the horizontal plane. With what velocity must another stone be projected vertically upwards that the two stones may rise to the same height above the horizontal plane?

(10) A particle is projected in a horizontal direction with a velocity of 10 miles an hour, and at the same time falls under the action of gravity. Assuming that no other forces are acting, and taking $g = 32$ (feet seconds), draw a picture representing the position of the particle at the end of 1, $1\frac{1}{2}$, $2\frac{1}{4}$, and 3 seconds.

(11) A stone is thrown into the air at an angle of 45° to the horizon, with a velocity of 128 feet per second. Show that the path of the stone will not be a straight line; and determine the amount of vertical deviation from a straight line at the end of 2 seconds, neglecting the resistance of the air.

(12) A gun is fired at an elevation of 45°; determine (1) the initial velocity, (2) the terminal velocity, (3) the greatest height, (4) the time of flight of the projectile—the distance from the gun at which it strikes the ground being given.

(13) A stone is let fall from the top of a railway-carriage which is travelling at the rate of 30 miles an hour. Find what horizontal distance the stone will have passed through in one-tenth of a second.

(14) Find the range of a projectile on a horizontal plane, being given the vertical and horizontal components of its initial velocity.

(15) A rifle is pointed horizontally with its barrel 5 feet above a lake. When discharged, the ball is found to strike the water 100 feet off. Find approximately the velocity of the ball.

(16) If any number of particles be projected from the same point at the same time, and with the same vertical component of the velocity of projection, prove that they will all ascend to the same height, and will reach their greatest height at the same moment.

CHAPTER VIII

(1) A mass originally at rest is acted on by a force which, in $\frac{1}{368}$ of a second, gives to it a velocity of $5\frac{1}{4}$ inches per second. Show what proportion the force bears to the weight of the mass.

(2) Two bodies of equal mass, each acted on by a uniform force, and each moving from rest, are observed to travel through the same space, the one in m seconds, the other in n seconds. Compare the forces acting on the two bodies, and prove the proposition upon which your result depends.

(3) A body of given mass is acted on by a constant force. Find the space described in a given time.

(4) If a particle move from rest through 40·5 feet in $4\frac{1}{2}$ seconds, under the action of a constant force, find the acceleration.

(5) A body of mass M, moving from rest under the action of a uniform force describes a space of one foot in the nth second. What is the force acting on the body?

(6) To each end of a string a weight of one kilogram is attached, and the string is passed over a smooth pulley. If a weight of 50 grams be added to one of the weights, determine the space through which that weight will descend in 10 seconds, the weight and rigidity of the string being neglected.

(7) When a body in motion is left wholly to itself, not being influenced by gravity or any other external force, what is the nature of the motion of translation of the body? Give some evidence from observation or experiment of the truth of your statement.

(8) Define the mass of a body, and describe an experiment which shows that the masses of bodies are proportional to their weights, giving fully the steps of the argument.

(9) The two boxes in Atwood's machine are so adjusted that one contains $3\frac{1}{2}$ and the other $2\frac{1}{2}$ ounces. How long will the heavier take to fall 1 foot? Also what will be the tension of the string during the motion?

(10) If a weight of 10 pounds on a smooth horizontal table be moved along the table by a force equivalent to 1 pound, find the acceleration produced.

(11) A cord without weight or friction passing round a single fixed pulley has a weight of 1,000 grains attached to one of its extremities, and one of 2,000 grains to the other. When left free, how far will this latter extremity have descended in 2 seconds?

(12) Two weights, of 5 and 7 pounds respectively, are fastened to the ends of a cord passing over a frictionless pulley supported by a hook. Show that, when they are free to move, the pull on the hook is equal to $11\frac{2}{3}$ pounds' weight.

(13) Two heavy bodies are connected by a flexible string which passes

over a fixed pulley. Show how to find the acceleration with which the heavier body will descend.

(14) Show how to use Atwood's machine to show (*a*) that a body acted on by a constant force moves with a uniform acceleration; (*b*) that the acceleration of a given mass is proportional to the force which acts upon it.

(15) Two scale-pans, each weighing 2 ounces, are suspended by a weightless string over a smooth pulley. A mass of 10 ounces is placed in one and 4 ounces in the other. Find the tension of the string and the pressure on each scale-pan.

(16) A certain force acting on a mass of 10 pounds for 5 seconds, produces in it a velocity of 100 feet per second. Compare the force with the weight of 1 pound, and find the acceleration it would produce if it acted on a ton.

(17) How would you determine the number of British absolute units of force in the weight of a pound?

(18) What difference in principle is there between the employment of a pair of scales and of a spring balance, and under what circumstances will the two methods lead to different practical results?

(19) A train of 200 tons is urged forward with a force equal to the weight of $1\frac{1}{3}$ tons, while the resistance it experiences is equal to the weight of 1 ton. What will be the measure of its acceleration, and how long will it take to acquire a velocity of 10 miles an hour?

(20) Two weights of 14 and 18 ounces are suspended by a fine thread which passes over a smooth pulley at a height of 30 feet above the weights. If the system be left free to move, find how far the heavier weight will descend in the first three seconds of its motion. Find the tension of the string.

(21) State your reason for regarding a pound as a unit of mass and not of force. What is the most convenient unit of force when a foot, a pound, and a second are units of length, mass, and time, respectively?

(22) A body resting on a smooth horizontal table is acted upon by a horizontal force equal to the weight of 2 ounces, and moves on the table over a distance of 10 feet in 2 seconds. Find the mass of the body.

(23) A mass of 488 grams is fastened to one end of a cord which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grams may rise through a height of 200 centimetres in 2 seconds? ($g = 980$.)

(24) The two ends of a string passing over the pulley of an Atwood's machine are loaded as follows: A with $16\frac{1}{2}$, B with $15\frac{1}{2}$ ounces. Find the tension at A when it is in motion downwards.

(25) A locomotive, 15 tons in weight, being supposed to acquire a velocity of 20 miles an hour in moving through a distance of a mile under the action of a constant difference of moving and resisting forces; calculate in pounds the requisite difference of the forces.

(26) Two masses of 48 and 50 grams respectively are attached to the string of an Atwood's machine, and, starting from rest, the larger mass passes through 10 centimetres in one second. Determine from these data the value of the acceleration due to gravity, your units being centimetres and seconds.

(27) I suddenly jump off a platform with a 20 pound weight in my hand. What will be the pressure of the weight upon my arm while I am in the air? Give a reason for your reply.

(28) In Atwood's machine, one of the boxes is heavier than the other by half an ounce. What must be the load of each in order that the overweighted box may fall through one foot during the first second?

(29) The intensity of gravity at the surface of the planet Jupiter being about 2·6 times as great as it is at the surface of the earth, find approximately the time which a heavy body would occupy in falling from a height of 167 feet to the surface of Jupiter.

(30) Force may be defined as 'any cause which tends to change a body's state of rest or of uniform motion in a straight line.' Mention any forces you know of, and show how this definition applies to them. What is *inertia*? Is it a force?

(31) The speed of a railway-train increases uniformly for the first 3 minutes after starting; and during this time it travels 1 mile. Supposing the line level, and disregarding friction and the resistance of the air, compare the force exerted by the engine with the weight of the train.

CHAPTER IX

(1) State Newton's Second Law of Motion, explaining the meaning of the terms employed.

(2) A particle of given mass describes one side of a regular polygon of n sides with a given uniform velocity. Determine the magnitude and direction of the momentum which must be given to it when it comes to the end of that side in order to make it describe the next side with the same velocity.

(3) While a railway train travels $\frac{1}{2}$ a mile on a level line, its speed increases uniformly from 15 miles an hour to 30 miles an hour; show what proportion the pull of the engine bears to the weight of the train (neglect friction).

(4) A particle, whose mass is M pounds, moves from rest under the action of a force of P units, which is constant in magnitude and direction. How far will the particle move in n seconds, and what space will it describe in the nth second?

(5) In the preceding example, if the force be the weight of the body, and the particle traverse 176·99 feet during the sixth second of its motion, find the value of 'g.'

(6) Deduce the parallelogram of forces from Newton's Second Law of Motion. To what other physical quantities does the parallelogram law apply?

(7) A cannon-shot of 1,000 pounds strikes a target directly with a velocity of 1,500 feet per second, and comes to rest; what is the measure of the impulse? If the cannon-shot rebounded with a velocity of 200 feet per second, what would be the measure of the impulse?

(8) Two bodies, whose masses are 31 ounces and 33 ounces respectively, suspended at the two ends of a thin string passing over a smooth pulley,

are allowed to move freely for 3 seconds. What momentum will be acquired by each body?

(9) Equal forces act for the same time upon bodies of different mass. What is the relation between the effects which they produce?

(10) Describe fully the unit of force implied in the equation $p = mf$.

(11) A body of mass m has been moved from rest by the action of a constant force P, through a space s, and has thereby acquired a velocity v. Prove that the following relations hold:

$$s = \frac{tv}{2}, \quad P = \frac{mv^2}{2s} = \frac{mv}{t}.$$

(12) Which could you throw farther, a small ball of lead, or a ball of cork of the same size? and why?

(13) A particle moves in consequence of the continued action upon it of a constant force. Show what is the character of the resulting motion, and in what manner it depends on the magnitude of the force and the mass of the particle. As a special case, show how the resulting motion would be changed if the mass of the particle were trebled, and the intensity of the force acting upon it were doubled.

CHAPTER X

(1) A weight of 1 pound is supported by two strings, of which one is inclined at an angle of 30°, and the other at an angle of 60° to the horizon. What is the tension of each of these strings?

(2) If several forces, which do not balance, act in the same plane upon different points of a solid body, one point of which is fixed, what condition must be fulfilled in order that the body may be in equilibrium? Show that this condition cannot be fulfilled unless the point is in the plane of the forces.

(3) A weight rests upon a smooth horizontal plane, and is acted on by a force equal to the weight of 6 pounds, in a direction inclined obliquely downwards at an angle of 30° to the horizon. Find the magnitude of the horizontal force required to prevent motion.

(4) A weight of 24 pounds is suspended by two flexible strings, one of which is horizontal, and the other inclined at an angle of 30° to the vertical direction. What is the tension in each string?

(5) If two forces acting at a point are represented in magnitude and direction by two sides of a triangle, under what circumstances will the third side correctly represent their resultant? Forces of 20 and 10 act along the sides A B and B C respectively of an equilateral triangle. Find the magnitude of their resultant.

(6) It is required to substitute for a given vertical force, two forces, one horizontal, the other inclined at an angle of 45° to the vertical. Determine the magnitude of these two forces.

(7) Show how to find the resultant of two forces, A and B, which act upon a given point in directions which make an angle α with each other, the force A acting towards the given point and the force B away from it.

(8) A boat is moored in a stream by two ropes, fixed to posts, one on each bank, and inclined to the direction of the current at angles of 30° and and 45°. Draw a figure from measuring which the proportion may be found between the strains on the two ropes.

(9) Three strings are tied in a knot; the ends of two of them are fastened to pegs, and the third has a known weight attached to it. Give a construction for finding the forces pulling the pegs; and from the construction show to what these forces respectively become almost equal, when one of the supporting strings is almost long enough to allow the other to hang in a vertical position.

(10) A B C D is a square; a force of 1 pound acts along the side A B from A to B; a force of 1 pound acts along the side A D from A to D; and a force of 2 pounds acts along the side C B from C to B. Determine the magnitude and position of the resultant of the three forces.

(11) Two men, A and B, carry a weight of 200 pounds on a pole between them. If the men be five feet apart, and the weight slung at a distance of 2 feet from A, what part of the weight will he bear, neglecting the weight of the pole?

(12) Two forces whose magnitudes are as 3 to 4 act at a point in directions at right angles, and produce a resultant of 2 pounds. Find the forces.

(13) A force of 20 pounds acts on a particle in a given direction. It is required to replace it by two other forces making respectively the angles 90° and 30° with this direction. Find these two forces.

(14) Two forces, equivalent to 36 pounds and 48 pounds, act at a point (i) in the same direction; (ii) in opposite directions; (iii) at right angles to each other. Compare their resultants in these three cases.

(15) A picture weighing 40 pounds is hung, with its upper and lower edges horizontal, by a cord fastened to its two upper corners and passing over a nail, so that the parts of the cord at the two sides of the nail make an angle of 60° with each other. Find the pull in the cord in pounds weight.

(16) Explain what is meant in mechanics by a *couple*, and by the *moment of a couple*. If two men are working a capstan, what conditions must be fulfilled in order that their action on the capstan may be represented by a couple? How would you estimate the moment of the couple exerted by them?

(17) State the conditions of equilibrium of any number of forces in a plane.

State and prove the conditions of equilibrium of three parallel forces.

(18) A force equal to the weight of 20 pounds, acting vertically upwards, is resolved into two forces, one of which is horizontal and equal to the weight of 10 pounds. What is the magnitude and direction of the other component?

(19) A man carries a bundle at the end of a stick over his shoulder. If the distance between his hand and the bundle be kept constant, and the distance between his hand and shoulder be changed, how does the force on his shoulder change?

(20) The horizontal and vertical components of a certain force are equal to the weights of 5 pounds and 12 pounds respectively. What is the magnitude of the force?

(21) Explain the triangle of forces, and illustrate its meaning and use by a practical case to which it may be applied.

(22) A particle is acted on by any number of forces in one plane. How would you express the conditions necessary for its equilibrium?

(23) Explain what will take place when three forces, which are represented by A B, B C, C A respectively, *act along* the sides of a triangular board A B C, which is supported on a smooth peg passing through its centre of gravity.

(24) What is the resultant of a system of forces? Is a system of forces in one plane always equivalent to a single force? If not, what exceptions may occur?

(25) Find a point inside a triangle, such that, if it be acted on by forces represented by the lines joining it to the vertices, it will be in equilibrium.

(26) Forces P, 2 P, 3 P and 4 P act along the sides of a square A B C D taken in order. Find the magnitude, direction, and line of action of the resultant.

(27) State the conditions for the equilibrium of a body which can only move in one plane, and which has one point fixed in that plane.

(28) Three forces act along three sides of a parallelogram A B D C, one from A to B, one from A to C and the third from B to D; each force being proportional to the side along which it acts; the parallelogram is such that the diagonal A D is perpendicular to the side B D. Find the line of action of the resultant force, and show that its magnitude is equal to one of the given forces.

(29) Show, by the aid of a sketch exhibiting the resolution of forces, how a ship can sail at right angles to the direction of the wind.

(30) Explain the action of the rudder in turning a vessel at sea.

(31) Three cords are tied together at a point. One of these is pulled in a northerly direction with a force of 6 pounds, and another in an easterly direction with a force of 8 pounds. With what force must the third cord be pulled in order to keep the whole at rest?

(32) Two parallel forces, P and Q, act at two points in a straight line, 6 inches apart, in opposite directions. Their resultant is a force of 1 pound acting at a point in the line 4 feet from the larger of the forces P and Q. Determine the values of P and Q.

(33) A particle is acted on by a force whose magnitude is unknown, but whose direction makes an angle of 60° with the horizon. The horizontal component of the force is known to be 1·35. Determine the total force, and also its vertical component.

(34) Prove that the sum of the moments of two forces acting in one plane about any point therein is equal to the moment of their resultant about the same point.

(35) A heavy plummet is immersed in a stream, the string being held by a person standing on the bank. The string is found to settle in a sloping position. Show by means of a sketch the three forces which keep the plummet in equilibrium.

(36) A weight of 4 pounds is suspended by a string, and is also acted on by a horizontal force. If, in the position of equilibrium, the tension of the string is 5 pounds, what is the horizontal force?

(37) Show how to resolve a given force into two components, one of which has a given magnitude and acts parallel to a given straight line.

(38) Show that the resultant of the forces 7 and 14 acting at an angle of 120°, is the same as the resultant of forces 7 and 7 acting at an angle of 60°.

39. When a horse is employed to tow a barge along a canal, the tow-rope is usually of considerable length; give a definite reason for using a long rope instead of a short one. Show whether the same considerations hold good in relation to the length of the rope when a steam-tug is used instead of a horse.

CHAPTER XI

(1) Assuming the principle of the Parallelogram of Forces, show what must be the relation, in order that there may be equilibrium on the inclined plane, between the power, the weight and the pressure against the plane.

(2) How may the same relation be deduced from the consideration of an endless chain put round the inclined plane and hanging down?

(3) Construct an inclined plane on which a horizontal force of 3 pounds will support a weight of 4 pounds, and find the resistance of the plane.

(4) A body whose mass is five kilograms, resting upon a smooth plane inclined at 30° to the horizon, is acted on by four forces; (i) its weight, (ii) the reaction of the plane, (iii) a force equal to the weight of two kilograms, acting parallel to the plane and upwards, and (iv) a force P acting at an angle of 30° to the plane. Determine the value of P that the body may be in equilibrium.

(5) A body weighing 6 pounds is placed on a smooth plane, which is inclined 30° to the horizon. Find the two directions in which a force equal to the weight of the body may act to produce equilibrium.

(6) In a system of 1 fixed and 4 movable pulleys, in which one end of each string is fixed to a beam, find the relation between the power and the weight (neglecting the weight of the pulleys), when one of the strings is nailed to the pulley round which it passes. What is the force exerted on the beam to which the strings are attached?

(7) A man, sitting on a board suspended from a single movable pulley, pulls downwards at one end of a rope which passes under the movable pulley and over a pulley fixed to a beam overhead, the other end of the rope being fixed to the same beam. What is the smallest proportion of his whole weight with which the man must pull in order to raise himself?

(8) In the previous example, with what force would the man require to pull upwards, if the rope, before coming to his hand, passed under a pulley fixed to the ground, as well as round the other two pulleys?

(9) Sketch a system of pulleys, one fixed and two movable, in which one end of the string passing round each pulley is attached to the weight; and find the relation of the power to the weight in equilibrium.

(10) When there is equilibrium in that system of pulleys in which one end of the string passing round each pulley is attached to a fixed support,

and the system is displaced, show that the power is to the weight as the space through which the weight is lifted is to the space through which the power is moved, the weights of the pulleys being neglected.

(11) If a man has to raise a weight, and has only one pulley at his disposal, show how he must apply it in order to obtain the utmost advantage.

(12) A man supports a weight equal to half his own weight by a number of pulleys of the second system, of which the upper block is attached to the ceiling. If there be seven strings at the lower block, find his pressure on the floor on which he stands.

(13) Describe and sketch a system of pulleys on which (neglecting the weight of the pulleys) a *power* of $10\frac{1}{2}$ pounds would balance a *weight* of 84 pounds, and show how far the power must move in order to raise the weight 3 feet.

(14) Find the relation between the power P and the weight W in a system of five movable pulleys, in which each pulley hangs by a separate string, and the weight of each pulley is equal to P.

(15) Describe the mechanical power called the First System of Pulleys, and show that if turned upside down it becomes the third system.

(16) Make a careful drawing of a system of weightless and frictionless pulleys in which a force equal to the weight of one pound can support a weight of 31 pounds.

(17) In a system of pulleys in which a separate string passes round each pulley, it is found that when the power and the weight are in equilibrium, and the power is caused to descend through 8 feet, the weight rises through 1 foot. Can the mechanical advantage of this arrangement ever be so much as 8? Give reasons for your answer.

(18) In a combination of one fixed and one movable pulley, where the power acts horizontally and the fixed end of the string is attached to the upper block, find the direction of the total resultant pressure on the upper block, and its magnitude as compared with the power.

(19) Ten weights, each of 20 pounds, are to be lifted to a height of 8 feet from the ground. Show how a system of pulleys might be arranged so that, disregarding friction and the weight of the pulleys, all the weights could be lifted together by exerting a force equal to one of them.

(20) Find the ratio of the power to the weight for equilibrium on a bent lever of the first kind, when the forces act at right angles to the arms. Supposing the arms to make an angle of 120° with each other, and have the relative lengths 1 and 5, find the magnitude and point of application of the resultant of the power and weight when the lever is in equilibrium.

(21) Where must the fulcrum be placed so that 2 pounds and 9 pounds may balance one another at the extremities of a straight lever 5 feet long?

(22) A weight is to be raised by means of a rope passing round a horizontal cylinder 10 inches in diameter, turned by a winch with an arm $2\frac{1}{2}$ feet long. Find the greatest weight which a man could so raise without exerting a pressure of more than 50 pounds on the handle of the winch.

(23) A wheel and axle is used to draw up a bucket of water from a well. Compare the mechanical advantage of the machine at first and after one coil of rope has been wound round the axle.

(24) Find the ratio between the power and the weight in a screw with 10 threads to the inch, worked by a lever 1 foot long.

(25) What is the ratio between the power and the weight in a screw which has 6 threads to the inch, and is moved by a power acting perpendicularly to an arm at a distance of $1\frac{1}{2}$ feet from the axis?

(26) A screw, whose pitch is $\frac{1}{4}$ inch, is turned by means of a lever 4 feet long. Find the power which will raise 15 cwt.

CHAPTER XII

(1) A weight of 6 ounces is drawn up along the lid of a smooth desk, which is 4 feet in length and rises 2 in 9, by a weight of 5 ounces, which hangs over the top of the desk (being attached to the first weight by a string), and descends vertically. Find the velocity acquired when the heavier body reaches the top of the desk.

(2) The last carriage of a railway-train gets loose while it is running at the rate of 30 miles an hour, up an incline of 1 in 150. Supposing the effect of friction on the motion of the carriage to be equivalent to a uniformly retarding force equal to $\frac{1}{300}$ of the weight of the carriage, find (i) the length of time during which the carriage will continue running up the incline, and (ii) the velocity with which it will be running down after the lapse of twice this interval from the instant of its getting loose.

(3) A particle slides down a smooth inclined plane under the action of gravity. If the particle start from rest, find the time of describing a given space.

(4) A particle slides from rest down any diameter of a vertical circle. Show that the velocity acquired in the descent varies inversely as the time of descent.

(5) A weight of 10 pounds resting without friction on a perfectly smooth horizontal plane, has a constantly acting horizontal pressure of 2 pounds applied to it. What will be its velocity after one second?

(6) A particle is projected up an inclined plane. Find the inclination that it may ascend a length of $9g$ feet and come to rest in 6 seconds, where g is the numerical measure of gravity.

(7) A train of 120 tons is to be taken from one station to another, a mile off, up an incline of 1 in 80, in 4 minutes, without using the brakes. Prove that, neglecting passive resistances, the engine must exert a pull, until steam is turned off, of about 6,203 pounds.

(8) A heavy body is placed on a smooth inclined plane whose length is 5 feet and height 3 feet. With what acceleration will it descend?

(9) A heavy body slides down a smooth plane inclined 30° to the horizon. Through how many feet will it fall in the fourth second of its motion?

(10) A mass of 19 pounds and a mass of 5 pounds are connected by a string which passes over a pulley at the edge of a horizontal table, so that the smaller mass hangs vertically, and, by its weight, pulls the larger mass along the table. Determine the acceleration, friction being neglected.

(11) If a ball slides without friction down an inclined plane, and in

the fifth second after starting, passes over 2207·25 centimetres, find its acceleration and the inclination of the plane to the horizon. Assume $g = 981$ (cm. sec.).

(12) The height of an inclined plane is $\frac{3}{5}$ of its length; a body is projected up the plane from the bottom with a velocity of 50 feet per second, and slides down again. Find the distance attained and the time before the body arrives at the starting point.

(13) Prove that the time taken by a particle to slide down any chord of a circle drawn from the highest point is constant, the circle being in a plane inclined to the vertical.

(14) A heavy body starting from rest slides down a smooth plane inclined 30° to the horizon. How many seconds will it occupy in sliding 240 feet down the plane, and what will be its velocity after traversing this distance?

(15) A smooth inclined plane, whose height is one half of its length, has a small pulley at the top, over which a string passes. To one end of the string is attached a mass of 12 pounds, which rests on the plane; while from the other end, which hangs vertically, is suspended a mass of 8 pounds; and the masses are left free to move. Find the acceleration and the distance traversed from rest by either mass in 5 seconds.

(16) A mass of 6 ounces slides down a smooth inclined plane whose height is half its length, and draws another mass from rest, over a distance of 3 feet in 5 seconds, along a horizontal table which is level with the top of the plane, the string passing over the top of the plane. Find the mass on the table.

(17) A particle projected up a smooth rectilinear groove, inclined at a given angle to the ground, being supposed to start with a given velocity from a given position; required, the extreme height to which it will ascend against the action of gravity, and the entire time which it will take to reach it.

(18) Divide an inclined plane of given length into two parts such that a body sliding from rest down the plane under the action of gravity may describe them in equal intervals of time.

(19) From a point in a smooth inclined plane a ball is rolled up the plane with a velocity of 16 feet per second. How far will it roll before it comes to rest, the inclination of the plane to the horizon being 30°? Also, how far will the ball be from the starting point after 5 seconds from the beginning of the motion?

(20) Find the tension of a rope which draws a carriage of 8 tons weight up a smooth incline of 1 in 5, and causes an increase of velocity of 3 feet per second in each second.

(21) If, on the same incline, the rope breaks when the carriage has a velocity of 48 feet per second, how far will the carriage continue to move up the incline?

CHAPTER XIII

(1) A weight of 56 pounds is attached to a straight lever without weight, at a distance of 3 inches from the fulcrum, and is balanced in one case by a power of 6 pounds, and in another case by a power of 16 pounds.

Find in each case the pressure on the fulcrum, when the power and weight are applied (*a*) on the same side of the fulcrum, (*b*) on opposite sides of the fulcrum.

(2) In the preceding example, show how the answer to each part would be affected if the lever weighed 9 pounds, and its centre of gravity were at the fulcrum.

(3) A solid roller, with an axle projecting from one end, is suspended horizontally by two vertical cords—one of them attached to the end of the roller opposite to the axle, the other to the middle of the axle: the roller is 4 feet long and weighs 27 pounds, the axle is 1 foot long and weighs 1 pound. Find the weight supported by each cord.

(4) Find the centre of gravity of three equal rods, A B, A C, and A D, in the same plane, and diverging from the point A, each of the angles B A C and C A D being one-third of a right angle.

(5) Two heavy particles, weighing respectively 3 and 5 ounces, are attached to the ends of a straight rod 8 inches long weighing 2 ounces. Find the centre of gravity of the system.

(6) A triangular slab is supported in the air by means of 3 vertical strings attached to its angular points. Prove that, in whatever way the weight of the slab is distributed, the tension on each string is independent of the position in which the slab is held, and that if the weight is distributed uniformly over the surface, the tension of each string will be one-third of the weight.

(7) A cylindrical vessel, weighing 4 pounds, and the internal depth of which is 6 inches, will just hold 2 pounds of water. If the centre of gravity of the vessel, when empty, is 3·39 inches from the top, determine the position of the centre of gravity of the vessel and its contents when full of water.

(8) Two uniform cylinders of the same material, one of them 8 inches long and 1 inch in diameter, the other 4 inches long and 2 inches in diameter, are joined together, end to end, so that their axes are in the same straight line. Find the centre of gravity of the combination.

(9) Define the centre of parallel forces. Three parallel forces act at the three vertices of a triangle; prove that the centre of the parallel forces cannot coincide with the centre of gravity of the triangle, unless the three forces are equal.

(10) A straight lever, 20 inches long, weighs 15 ounces. Where must the fulcrum be placed in order that the lever may be in equilibrium when a weight of 16 ounces is hung at one end, and a weight of 9 ounces at the other?

(11) A piece of uniform paper in the form of a regular hexagon has one of the equilateral triangles, obtained by joining the centre to two consecutive angular points, cut away. Determine the position of the centre of gravity of the remainder of the paper.

(12) A parallelogram is divided along a diagonal; and one half remaining fixed, the other half is lifted, reversed, and applied to the former half along the same diagonal. Find the distance between centres of gravity of the quadrilateral figure thus formed and of the original parallelogram.

(13) A straight lever, 6 feet long and heavier towards one end, is found

to balance on a fulcrum 2 feet from the heavy end; but when placed on a fulcrum at the middle, it requires a weight of 3 pounds, hung at the lighter end, to keep it horizontal. What is the 'weight' of the lever?

(14) Find the centre of gravity of a system consisting of two spheres, 8 ounces and 24 ounces in weight, connected by a rigid rod withou weight, the distance between the centres of the spheres being 1 foot.

(15) Prove that for every rigid body there exists one, and only one, point which has the property that, if it be supported, the body will rest in any position. Where is the point situate in the case of a spherical shell? and in what sense is the proposition true in this case?

(16) Show how to find the centre of gravity of a system of heavy particles.

(17) Weights are placed at the corners of a triangle, each weight being proportional to the length of the opposite side; show that the centre of gravity of the weights is situated at the centre of the circle inscribed in the triangle.

(18) Show how to find the centre of gravity of a body when the body is made up of two parts, and the centre of gravity of each part is known.

(19) Three equal rods are jointed together so as to form an equilateral triangle A B C. If D, E be the middle points of the rods A B, A C, prove that the distance between the centres of gravity of the portions D A E, E C B D, of the rods is $\frac{5}{8}$ of the altitude of the triangle.

(20) A square is divided into four equal squares by straight lines drawn parallel to its sides; if one of the four squares be removed, find the centre of gravity of the remaining figure.

(21) Find the centre of gravity of the frustrum of a right cone, having given the radii of the two circular ends and the altitude of the frustrum.

(22) A right-angled isosceles triangle is cut out of a given circular disc, the angular points being on the circumference. Find the distance of the centre of gravity of the remainder from the centre of the circle.

(23) A B C is an equilateral triangle of 6 inches side, of which O is the centre. If the triangle O B C be removed, find the distance from A to the centre of gravity of the remainder.

(24) A uniform rod A B is 4 feet long and weighs 3 pounds. One pound is then attached to the end A, 2 pounds at a point distant 1 foot from A, 4 pounds at 3 feet from A, 3 pounds at 2 feet from A, and 5 pounds at the end B. Find the distance from A of the centre of gravity of the system.

(25) A cylindrical vessel 1 foot in diameter and 1 foot high, open at the top, is made of thin sheet metal of uniform thickness. Find the height of the centre of gravity from the bottom of the vessel. If the vessel be half-filled with water, where will the common centre of gravity of the vessel and the water be, assuming that the weight of the vessel is one-fifth that of the water in it.

(26) A heavy uniform beam, 10 feet long, whose mass is 10 pounds, is supported at a point 4 feet from one end; at this end a mass of 6 pounds is placed; find the mass required at the other end to balance the beam.

(27) Weights of 1 pound, 2 pounds, 3 pounds, and 4 pounds, are suspended from a uniform lever 5 feet long at distances of 1 foot, 2 feet, 3 feet, and 4 feet respectively from one end. If the mass of the lever is 4 pounds, find the position of the point about which it will balance.

(28) A solid right circular cone of homogeneous iron is 64 inches in height, and its mass is 8,192 pounds. The cone is cut by a plane perpendicular to the axis, so that the mass of the small cone removed is 512 pounds. Find the height of the centre of gravity of the truncated portion remaining above the base of the cone.

(29) Prove that a body placed on a plane will stand or fall according as the vertical line through the centre of gravity does or does not fall within the base.

(30) A uniform plate of metal 10 inches square has a hole 3 inches square cut out of it, the centre of the hole being $2\frac{1}{2}$ inches distant from the centre of the plate. Find the position of the centre of gravity of the plate.

(31) A B C D E is a board of irregular figure; and it is found that when the board is hung from A, the point C is in the vertical line through A; and when it is hung from B, the point D is in the vertical line through B. If the board be suspended from the point E, find what point in the perimeter will lie vertically below E.

(32) Draw a four-sided figure with unequal sides, and describe (i) a geometrical, and (ii) an experimental, method of finding its centre of gravity.

(33) A heavy equilateral triangle is placed with its plane vertical, and one side resting on a rough inclined plane; the coefficient of friction being $\sqrt{3}$. What is the greatest inclination of the plane to the horizon that the triangle may neither slide down the plane nor roll over an angular point?

(34) A uniform equilateral triangle has a sphere, of the same weight as the triangle, attached to it, so that the centre of the sphere is at one angular point of the triangle. If the triangle be suspended by a string attached to the middle point of one of the sides, which passes through the centre of the sphere, show how to determine the inclinations of the sides of the triangle to the string in equilibrium.

(35) A cylinder is placed on one of its ends on a horizontal plane sufficiently rough to prevent sliding, and is gradually pulled by a string attached to its upper end until it falls over. What part of the upper surface is just over the point of contact of the base with the plane when this takes place?

(36) Show from the property of the centre of gravity that in a common balance it makes no difference in what part of the scale-pan the weight is put, whether in the centre or at the edge.

(37) A uniform triangular lamina, whose sides are 3, 4, and 5 inches respectively, is suspended by a string from the middle point of the longest side. Draw a figure showing clearly the position of the opposite angular point. If equal weights be attached to the three angular points, how will the position of equilibrium be affected?

(38) A trapezium, having two parallel sides, which are 4 and 12 feet long, and the other sides each equal to 5 feet, is placed with its plane vertical and its shortest side on an inclined plane. Find the relation between the height and base of the plane when the trapezium is on the point of falling over.

(39) Weights are attached to a series of points along a weightless rod. Show that the rod, if supported at a point so as to rest in a horizontal position, will also rest in any other position.

(40) What conditions must be secured to ensure accuracy in a beam-balance, and how can the sensibility of such a balance be varied?

(41) A body appears to weigh 24 pounds when placed in one scale-pan and 25 pounds when placed in the other. Find its real weight to three places of decimals.

(42) A piece of lead placed in one pan A of a balance is counterpoised by 100 grams in the other pan B. When the same piece of lead is placed in the pan B, it requires 104 grams in the pan A to balance it. Show what is the ratio of the lengths of the arms of the balance.

CHAPTER XIV

(1) A pole 12 feet long, weighing 25 pounds, rests with one end against the foot of a wall, and from a point 2 feet from the other end a cord runs horizontally to a point in the wall 8 feet from the ground. Find the tension of the cord and the pressure on the lower end of the pole.

(2) A rod A B, 5 feet long, without weight, is hung from a point C by two strings which are attached to its ends and to the point; the string A C is 3 feet, and B C is 4 feet in length; and a weight of $2\frac{1}{2}$ pounds is hung from A, and an equal weight from B. Find the tensions of the strings in the condition of equilibrium.

(3) The extremities of the horizontal diameter of a circular disc, weighing 6 ounces, are nailed against a wall, and to a point in the edge of the disc, at $\frac{1}{12}$ of the whole circumference from one of the nails, a weight of 4 ounces is attached. Find the pressure upon each nail.

(4) A lever consists of a uniform bar of weight w and length z; it is bent at an angle of 120° at its fulcrum, round which it can revolve in a vertical plane. If, when the lever is left to itself, the shorter arm is horizontal, what is the length of each arm? and what weight must be suspended from the extremity of the shorter arm in order that the longer arm may become horizontal?

(5) To each end of a uniform rod, 100 inches long and weighing 12 pounds, is fastened one end of a flexible string 140 inches long, to which a weight of 9 pounds is attached at a point 60 inches from one end. In what position will the rod remain in equilibrium about a pivot through the middle? and where must the pivot be placed in order that the rod may be balanced when horizontal?

(6) A man in the act of being weighed in an ordinary balance pushes with a walking-stick the beam of the balance at a point between the point of suspension of the scale in which he is and the fulcrum. What effect, if any, will be produced on his apparent weight?

(7) In the preceding example, if the scale in which the man is be kept from moving laterally by a horizontal string attached to a fixed point, what will be the effect?

(8) Two weights, W and W′, are carried on a pole which rests on the shoulders of two men, A and B, of equal height. The weight W rests at a point C, such that A C : C B = 3 : 2, and W′ at a point D, such that A D : D B = 5 : 2. Find the proportions of the weights borne by the two men.

(9) One end, A, of a uniform heavy rod rests on a rough horizontal floor, the coefficient of friction being μ, and the other, B, against the vertical face of a wall, the coefficient of friction being μ'. The vertical plane through the rod being perpendicular to the wall, find the limiting position of equilibrium. If, when the rod is in the position of equilibrium, we suppose μ to be infinite, find how far the end B may be moved along the face of the wall before the rod begins to slip.

(10) A triangular table has a leg at each angular point and rests on a rough floor, the coefficient of friction being μ. The table is acted on by a couple whose plane is horizontal, and is just on the point of motion. Find the directions in which the frictions at the three legs act, and determine whether all the legs will begin to move at the same time or not.

(11) Weights of 5, 6, 9 and 7 pounds respectively are hung from the corners, A, B, C, D, of a horizontal square, 27 inches in the side; find, by taking moments about two adjacent edges of the square, the point where a single force must be applied to the square to balance the effect of the forces at the corners.

(12) The arms of a bent lever are at right angles to one another, and their lengths are in the ratio of 5 to 1. The longer arm is inclined 45° to the horizon, and carries at its extremity a weight of 10 pounds. The end of the shorter arm presses against a smooth horizontal plane. Draw a figure showing the forces in action, and find the pressure between the shorter arm and the plane.

(13) A heavy uniform ladder rests with its upper end pressing against a smooth vertical wall. Show by a figure how to determine the direction of the resultant force acting upon the foot of the ladder.

(14) A uniform rod of given length is to be supported in a given inclined position with its upper end resting against a smooth vertical wall by means of a string attached to the lower end of the rod and to a point of the wall. Find by a geometrical construction the point of the wall to which the string must be attached.

(15) A uniform sphere rests on a smooth inclined plane, and is supported by a horizontal string. To what point on the surface of the sphere must the string be attached?

(16) A beam whose centre of mass divides it into two segments, A and B, is placed inside a smooth sphere. Find the position of equilibrium.

(17) A frame, consisting of two pairs of parallel bars hinged without friction at their four intersections, being supposed in strained equilibrium under the tensions of two diagonal ties; determine, given all particulars, the ratio of the tensions on the ties.

(18) A uniform bar, suspended from a fixed point by two imponderable cords attached to its extremities, being supposed in equilibrium under the action of gravity; determine, given all particulars, the ratios of the tensions on the cords to the weight of its mass.

(19) A ladder rests against the side of a house and is inclined at 60° to the ground. The pressure of the ladder against the wall being equal to a force of 60, and the friction at the same place equal to a force of 40, find the *pressure* and *friction* at the point where the ladder rests on the ground. Find also the *weight* of the ladder.

(20) Four pegs are fixed in a wall at the four highest vertices of a

regular hexagon (the two lowest being in a horizontal straight line), and over these is thrown a loop suspending a weight. The loop has such a length that the angles formed by it at the lowest pegs are right. Determine the tension of the string and the pressure on the pegs.

(21) A sphere of wood, loaded at one point with lead, rests upon a plane inclined at 30° to the horizon, being prevented from sliding down by the friction of the plane. State and explain by a diagram the conditions of equilibrium.

(22) A person ascends a ladder resting on a rough horizontal floor against a smooth vertical wall; determine, graphically or otherwise, the direction and magnitude of the force with which the ladder presses against the floor.

(23) A weight W, constrained to move in the circumference of a vertical circle, is attached to another weight P, hanging vertically by means of a cord passing over a fixed pulley at a given point on the circle. Find the position of equilibrium of W.

(24) State the mechanical conditions of equilibrium, when a beam of uniform thickness and density rests with one extremity against a smooth vertical wall, and the other against the inner surface of a smooth hemispherical bowl.

(25) A uniform circular arc rests in a vertical plane on two smooth fixed pegs. Find, by the aid of a sketch, the mechanical conditions of equilibrium.

(26) Two men carry a block of iron, weighing 176 pounds, suspended from a uniform pole 14 feet long and weighing 22 pounds; each man's shoulder is 1 foot 6 inches from his end of the pole. At what point of the pole must the heavy weight be suspended, in order that one of the men may bear $\frac{4}{5}$ of the weight borne by the other?

(27) If three forces acting on a particle in directions mutually at right angles are 20 pounds, 30 pounds and 60 pounds in magnitude, what is the magnitude of their resultant?

(28) A body of any form is supported by a string, which, passing over a smooth peg, is fastened at its extremities to two points in the body. Show that in its position of equilibrium the centre of gravity of the body is vertically beneath the peg, and that the two portions of the string make equal angles with the vertical direction.

(29) A bar of uniform thickness and density, 12 feet long, and of 1 cwt., is supported at its extremities in a horizontal position. If a body of 2 cwt. be suspended from a point distant 2 feet from one end, and a body of 4 cwt. at 4 feet from the other end, find the pressure on the points of support.

CHAPTER XV

(1) Four material particles of the relative masses 2, 3, 4, and 5, are placed at the corners A, B, C, and D respectively of a square. Find their common centre of mass.

(2) Two particles, of masses m and m', set out from the same point O, and travel with uniform velocities, v and v', along straight lines O A, O A',

inclined to one another at an angle α. Prove that the mass-centre of the two particles will move along a straight line with a uniform velocity.

(3) A particle, whose mass is equal to that of an equilateral triangle, is attached to one of the angular points. Find the mass-centre of the combination.

(4) Two inclined planes being placed back to back are at right angles to one another. Show that if two equal bodies starting contemporaneously from the right angle slide down the planes, their centre of mass will descend in a vertical line.

(5) Two heavy particles connected by a string, slide on two inclined planes in a vertical plane perpendicular to the common edge of the two planes. Prove that their centre of mass moves in a straight line.

(6) Explain the Third Law of Motion. Apply it to determine the velocity gained per second when weights of 6 ounces and 4 ounces are attached to the two ends of a string passing over the edge of a smooth table, the larger weight being drawn along the table by the smaller, which descends vertically.

(7) Give a numerical example of the Third Law of Motion. Explain the kick of a gun.

(8) A half-ton shot is discharged from an eighty-one-ton gun with a velocity of 1,620 feet per second. What will be the velocity with which the gun will recoil, if the mass of the powder be neglected?

(9) Equal spherical inelastic balls are placed at short equal intervals in a smooth horizontal groove; the first is projected from the end along the groove with a velocity of 20 feet per second. Find the velocities after successive impacts.

(10) An equilateral triangle is placed with one side vertical; and two equal masses, connected by a slack inelastic string, which passes without friction over the upper corner, are allowed to fall from the upper angle, the first down the slant side, the other down the vertical side. What start in time must be given to the first mass, that, when the string is pulled tight, the masses may destroy each other's velocity?

(11) A perfectly elastic particle P is projected from a point within a triangle, and, after striking and rebounding from the three sides successively, returns to the same point again. Supposing it to be then moving in the direction of projection, find the angles of the triangle described by P.

(12) An elastic ball is dropped from a height h, on a horizontal plane. Being given e, the coefficient of elasticity, find the height to which the ball will ascend after the nth rebound.

(13) The first of a row of balls of equal mass and perfect elasticity, ranged at any intervals along a smooth horizontal groove, being supposed projected with any velocity along the groove; required, the velocities of the several balls after the last of the resulting collisions.

(14) An elastic ball, moving vertically under the action of gravity, being supposed to fall through a height h upon a horizontal plane; required its coefficient of elasticity, in order that it may ascend again to height k above it.

(15) A sphere moving with a velocity v impinges directly on another

of twice its mass. Find the velocities after impact (*a*) if the two spheres are inelastic, (*b*) if they are perfectly elastic.

(16) Two spheres, of masses m and n, moving in the same straight line with velocities u and v, being supposed to interchange velocities by direct collision with each other; required the ratio of their masses, and coefficient of elasticity, in order that such interchange may take place.

(17) An imperfectly elastic ball impinges upon a smooth plane at an angle of 30°, and is deflected at an angle of 60° from the perpendicular. Find the coefficient of elasticity.

(18) A sphere, moving with a given velocity in a given direction impinges directly upon an equal sphere at rest. Determine the subsequent motion in the two cases in which the spheres are (*a*) perfectly inelastic, and (*b*) perfectly elastic.

(19) Two spheres of unequal weights are moving with equal velocities along the same straight line in opposite directions; after impact one of them is reduced to rest. Supposing the weights of the spheres to be given, deduce the coefficient of elasticity, and show that in no case can one of the spheres be more than three times the weight of the other. Is it the heavier or the lighter of the two spheres which is reduced to rest?

CHAPTER XVI

(1) How is the energy of a moving body estimated? Through what distance must a force equal to the weight of $\frac{1}{2}$-pound act upon a mass of 48·3 pounds in order to increase its velocity from 24 to 36 feet per second?

(2) When a freely-moving particle is acted on by a force, show that the gain of kinetic energy is equal to the work done by the force.

(3) A cannon-ball whose mass is 60 pounds falls through a vertical height of 400 feet. What is its energy? With what velocity must such a cannon-ball be projected from a cannon to have initially an equal energy?

(4) Prove that when a heavy body moves down a smooth inclined plane, the difference between the square of the velocity at a higher and a lower point depends only on the *vertical* distance between the two. Is the same proposition applicable to the case of projectiles in vacuo?

(5) A weight of W pounds is drawn from rest up a smooth inclined plane of height h and length l, by means of a string passing over a pulley at the top of the plane and supporting a weight of w pounds, hanging freely. Prove that in order that W may just reach the top of the plane, w must be detached after it has descended a distance

$$\frac{W + w}{w} \cdot \frac{h\,l}{h + l}.$$

(6) Equal forces act for the same time upon unequal masses M and m. What is the relation between (*a*) the momenta generated by the forces, (*b*) the amounts of work done by them?

(7) Find the kinetic energy in ergs of a cannon-ball of 10,000 grams discharged with a velocity of 50,000 centimetres per second.

(8) A shot of 1,000 pounds, moving at 1,600 feet per second, strikes a

fixed target. How far will the shot penetrate the target, exerting upon it an average pressure equal to the weight of 12,000 tons?

(9) A bullet, fired horizontally from a musket, being supposed to pass perpendicularly through a target suspended freely in the air; explain, on the theory of dynamical work, why the greater the velocity of the bullet the less the displacement of the target.

(10) A body whose mass is 100 grams is thrown vertically upwards with a velocity of 980 centimetres per second. What is the energy of the body (*a*) at the moment of propulsion, (*b*) after half a second, (*c*) after one second. ($g = 980$.)

(11) A ball weighing 10 pounds is projected vertically upwards with an initial velocity of 1,600 feet per second. Find its velocity, its momentum, and its potential energy; (*a*) after 30, and (*b*) after 60 seconds.

(12) A ball weighing 5 ounces and moving with a velocity of 1,000 feet per second, strikes a shield, and after piercing it moves on with a velocity of 400 feet per second. How much energy has been expended in piercing the shield?

(13) Show that when two perfectly elastic spheres collide directly, their kinetic energy will be unaltered by the collision.

(14) A train weighing 120 tons runs on a level road, and the resistances to be overcome are 8 pounds per ton. How many absolute units of work must be expended in making a run of 40 miles, when there is no useless expenditure of steam?

(15) How many units of work must be expended in raising from the ground the materials for building a uniform column 66 feet 8 inches high, and 21 feet square—a cubic foot of brickwork weighing one hundred-weight?

(16) Enunciate precisely the law of 'virtual work,' and apply it to determine the conditions of equilibrium in the single movable pulley with inclined strings, and in the screw.

(17) A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 inches, and while it makes 7 revolutions, the bucket, which weighs 30 pounds, rises $5\frac{1}{2}$ feet. Show what is the smallest force that can be employed to turn the wheel. Upon what general principle is your answer founded?

(18) Find the horse-power of an engine which is taking a train of 120 tons down an incline of 1 in 224 at 50 miles an hour, supposing a resistance of 35 pounds a ton due to friction on this incline.

(19) Find the horse-power of an engine which is drawing a train of 200 tons up an incline of 1 in 112 at 20 miles an hour; resistance on the level 8 pounds a ton.

(20) What is the horse-power of an engine which can project 10,000 pounds of water per minute with a velocity of 80 feet per second; 20 per cent. of the whole work being wasted by friction, &c.?

CHAPTER XVII

(1) Prove that the central acceleration of a body describing a circle of radius r with velocity v is v^2/r.

(2) A body of mass m moves in a circle to whose centre it is fixed by a string of length r. If the body make n revolutions in a second, what will be the tension of the string?

(3) Find the horizontal pressure on the rails exerted by an engine of 20 tons going round a curve of 600 yards radius at 30 miles an hour.

(4) A locomotive 15 tons in weight moves with a velocity of 20 miles an hour in a circle of one mile radius. Find in tons weight the horizontal pressure against the rails.

(5) A locomotive 20 tons in weight moves with a velocity of 30 miles an hour in a circle of 10 miles radius; find the pressure against the inner surface of the rails.

(6) A locomotive engine of 10 tons weight passes on level ground round a curve 600 yards in radius at the rate of 30 miles an hour. Determine the horizontal pressure on the rails.

(7) A body attached to one end of a string of length l, the other end of which is fixed, describes uniformly a horizontal circle, the string being constantly inclined at an angle α to the vertical. Prove that the time of a complete revolution is

$$2\pi\sqrt{\left(\frac{l\cos\alpha}{g}\right)}.$$

(8) If a small ring be constrained to describe a circle in a vertical plane under the action of gravity, prove that, if the particle make a complete circuit, the pressure at the bottom will be at least six or five times the weight of the ring, according as the ring is attached to the centre of the circle by a string, or is constrained to slide along a circular wire.

(9) There are two bodies, A and B, each of which is capable of turning about a fixed axis. A is acted upon for a time t_1 by forces whose moment about its fixed axis is M_1, and B is acted upon for a time t_2 by forces whose moment about its fixed axis is M_2. If the angular velocity of A is thus changed from ω_1 to ω_1', and that of B from ω_2 to ω_2', compare the moments of inertia (K_1, K_2) of A and B.

(10) In the preceding example, compare the kinetic energy gained by the first body with that gained by the second.

(11) Particles whose masses are 5, 7, 3 and 8 grams respectively are fixed at the corners of a rigid immaterial square, whose side is 10 centimetres; find the moment of inertia of the system about an axis through the centre of the square and perpendicular to its plane.

(12) Show that the moment of inertia of a uniform circular hoop about an axis through its centre perpendicular to its plane is equal to the mass of the hoop multiplied by the square of its radius.

(13) When a body moves uniformly in a circle, show that the centrifugal force varies directly as the radius, and inversely as the square of the periodic time.

ANSWERS TO THE EXAMPLES

SECTION 9 (p. 6)

(2) 1·1 feet. (3) $mq : pn$. (4) 25 feet per second; 15 feet per second.

SECTION 13 (p. 10)

(2) ·01 second. (3) 60 feet per second.

CHAPTER I

(1) About $17\frac{1}{2}$. (2) $\frac{1}{36}$ cm. per second. (3) $\frac{5}{18}$. (4) $\frac{25}{27}$. (5) 20 kilometres per hour. (6) 5·86 seconds. (7) f. (8) 10 cm. per second.

SECTION 16 (p. 14)

	Displacement	Velocity	Acceleration
(1)	+	−	−
(2)	−	+	−
(3)	−	−	+
(4)	0	−	0
(5)	0	0	−
(6)	+	0	+

SECTION 22 (pp. 18, 19)

(3) ~~−1 cm.~~ per second per second; ~~−38 cm.~~ per second; the direction of the initial velocity being taken as positive. (4) −6 feet per second per second.

SECTION 24 (p. 20)

(3) 30 feet per second. (4) 6 metres per second per second.

SECTION 25 (pp. 21, 22)

(3) 24·5 metres from the top of the tower, after $\sqrt{5}$ seconds.

CHAPTER II

(1) 2,450 cm. per second. (2) ·0132 second. (3) The velocity of the second stone is twice that of the first. (4) 200 cm. per second per second. (5) Another 32 seconds; 360 metres. (6) 4·76 metres per second

per second. **(7)** After another $4\frac{1}{2}$ seconds. **(8)** The first projected ball after a further interval of $3+3\sqrt{2}$ seconds, the second after $9+3\sqrt{2}$ seconds. **(9)** $n=8\cdot5$. **(10)** $(6+3\sqrt{2})\,g$; $(3+3\sqrt{2})\,g$. **(12)** *See errata.* **(13)** -4 metres per second per second. **(14)** $5\frac{5}{11}$ seconds. **(15)** 35,280. **(16)** ·05 cm. per second.

CHAPTER III

(1) $\dfrac{1}{360}$; $\dfrac{1}{2\pi}$. **(2)** Between 90° and 180°, or between 270° and 360°. **(3)** $-90°$. **(5)** $-1/\sqrt{2}$; -1. **(6)** $\sqrt{3}/2$. **(8)** 210° or 330°, &c. **(9)** 135° or 315°, &c. **(10)** 30°, 150°, 210°, 330°, &c. **(12)** The sum of the angles $= +90°$.

CHAPTER IV

(1) 1 metre; -1 metre per second per second. **(2)** 4 metres. **(4)** (*a*) By the 'time'-axis. (*b*) By a straight line parallel to the time-axis. **(5)** A velocity of 6,000 cm. per second; a motion whose acceleration varied at a uniform rate. **(6)** (*a*) Rest; (*b*) Uniform velocity.

CHAPTER V

(1) $-5\sqrt{3}$ metres. **(2)** About 4,000 miles. **(6)** $\sqrt{39}$ miles. **(7)** $\dfrac{a}{2}\sqrt{(121\pi^2-44\pi+8)}$, where a is the radius of the wheel. **(8)** $5\sqrt{(10-4\sqrt{2})}$ miles. **(9)** Southerly component 12 metres; south-easterly component $-12\sqrt{2}$ metres.

SECTION 75 (p. 72)

(1) $\dfrac{g}{2}$. **(2)** A horizontal acceleration of magnitude g.

CHAPTER VI

(3) $g\sqrt{2}$ in an upward direction inclined 45° to the vertical. **(4)** $250\sqrt{10}$ cm. per second per second. **(5)** $\sqrt{7}v$, where v is the smaller of the given velocities. **(6)** $\sqrt{30}$ metres per second. **(7)** $2\sqrt{2}$ cm. per second per second. **(8)** 25; $25\sqrt{3}$. **(9)** $\sqrt{3}f$ perpendicular to the direction of f. **(10)** $\sqrt{5}$ metres per second; $2\sqrt{2}$ m. per second; $2\sqrt{5}$ m. per second. **(11)** Equal to the initial velocity, and inclined to it 120°. **(12)** $\frac{1}{2}g$ backward.

CHAPTER VII

(1) 81·6 metres. **(2)** $2uv/g$, where u and v are the horizontal and vertical components. **(3)** After $\sqrt{2}$ seconds; $9\cdot8\sqrt{5}$ metres. **(5)** $1 : \sqrt{3}$. **(6)** Vertical distance from point of impact = 3,920 cm.; horizontal distance $= 5{,}880\sqrt{3}$ cm.; vertical velocity = 980 cm. per second; horizontal velocity

$=2{,}940\sqrt{3}$ cm. per second. **(7)** 12,262·5 cm.; 981 cm. per second per second. **(8)** $v\sqrt{2}$; $\sqrt{5}\,v^2/2g$. **(9)** Horizontal component 8 metres per second; vertical component 24·5 metres per second. **(10)** 8·72 metres.

SECTION 105 (pp. 98, 99)

(3) $\frac{1}{8}g$. **(4)** 50 g dynes; $33\frac{1}{3}$ grams. **(5)** 980 grams.

CHAPTER VIII

(1) 27·$\dot{7}$. **(2)** 18 grams and 2 grams. **(4)** The relative acceleration will change from $2\,\dfrac{m-m'}{m+m'}\cdot g$ to zero. **(5)** $39\frac{3}{8}$ dynes. **(6)** $163\frac{1}{3}$ grams. **(7)** ·98 cm. per second per second. **(10)** m^2/m'.

CHAPTER IX

(1) 24,000,000 (C.G.S.). **(2)** 3 : 2. **(3)** $k\,m\,v/n$. **(4)** $p\,t$. **(5)** 2 minutes; westward, 600,000 C.G.S. units. **(6)** 6×10^{10} dynes. **(7)** About 61,224 kilograms.

SECTION 117 (p. 108)

(1) 11,001 dynes; 1,100·1 . . . cm. per second per second. **(4)** 6,242 (C.G.S.).

CHAPTER X

(1) 5,000 dynes. **(3)** 120°. **(6)** $\frac{1}{2}a\,(p+q+r+s)$. **(8)** Through a point which lies on the straight line joining the corners 2 and 5, and is $\frac{5}{7}$ of a side from the latter corner. **(9)** Line of action parallel to A B and distant $M'a/\sqrt{2}\,(M-M')$ from C; force $=\sqrt{2}\,(M-M')/a$. **(10)** 14·17 poundals. **(11)** Shorter part, 80 grams' weight; longer part, 60 grams' weight. **(13)** 2·4 metres from that support. **(14)** 5 and 12 at right angles to one another; 13 making with twelve an obtuse angle whose sine is $\frac{12}{13}$. **(15)** 10 inches from the greater mass.

CHAPTER XI

(1) (*a*) $W/\sqrt{3}$; (*b*) $W/2$. **(2)** $W/(\sqrt{3}+2)$. **(3)** 7 : 4. **(4)** 3 : 7. **(5)** $W+w$. **(6)** 280 : 19. **(7)** Half his weight. **(8)** One-fifth of his weight (neglecting the weight of the pulleys). **(9)** 30 cm. from the end where the 'weight' acts. **(10)** The masses must be equal. **(11)** $\frac{1}{4}$ P. **(14)** 2 metres; $\sqrt{5}$ metres. **(15)** 12. **(16)** *See errata.* **(17)** 16. **(18)** 2π millimetres. **(19)** $\sqrt{3}$. **(20)** $1/\tan\theta$.

CHAPTER XII

(3) $1/\sqrt{3}$. **(4)** m'/m. **(5)** Acceleration

$$=\frac{(m_1\sin\alpha_1-m_2\sin\alpha_2\mp m_1\mu_1\cos\alpha_1\mp m_2\mu_2\cos\alpha_2)\,g}{m+m'};$$

pull of the string $= \frac{m_1 m_2}{m_1 + m_2}$ $(\sin \alpha_1 + \sin \alpha_2 \mp \mu_1 \cos \alpha_1 \pm \mu_2 \cos \alpha_2)$; the upper signs being taken throughout when m_1 is descending, and the lower signs in the contrary case. (6) $\sqrt{2}\,\mu\,W/(1+\mu)$. (7) $\sqrt{2}\,v$, in a direction which bisects internally the angle between the two parts of the string; where v is the velocity of either mass. (8) $\frac{v}{\sqrt{2}}$ in a direction which bisects externally the angle between the two parts of the string. (9) $v^2/4g$. (10) $\sqrt{2}-1$.

CHAPTER XIII

(1) Half-way between the sides 1, 4 and 3, 2, and $\frac{2}{5}$ of a side distant from the side 4, 2. (2) At a distance $(a^2+ab)/(2a+b)$ from the fourth side b. (3) $\frac{6}{11}\sqrt{2}$ of an edge from the eighth edge. (4) $\frac{3}{5}$ of an edge from the sixth face. (8) $(\sqrt{3}+1)\,a/(8+2\sqrt{3})$ of a side of the square from the centre of the square. (9) $\frac{5}{9}$ of the way from A to the middle point of B C. (10) At a distance from the centre of the larger circle equal to one-sixth of its radius. (12) $\sqrt{6}:1$. (13) 5 cm. (14) The centre of gravity lies in the horizontal plane passing through the axis. (15) 1,052·6 grams. (17) $n-n' : m-m'$. (18) $150m/(3m+2)$ cm. from the centre of gravity.

CHAPTER XIV

(1) $1/2\sqrt{3}$. (2) $\frac{1}{2}W \sin\theta$, where W is the weight of the wheel. (6) $10\sqrt{3}/3$ cm. from the centre of the ring.

CHAPTER XV

(1) At a distance from the centre of the rod equal to $\frac{1}{12}$ of its length. (6) $\frac{5}{24}$ of the unit length. (7) $m_1 m_2 v/(m_1+m_2)$. (8) $1+e : 1+e'$. (11) $1/\sqrt{2}$.

CHAPTER XVI

(1) 49×10^{10} ergs. (2) $\frac{1}{2}m\,(gl-v^2)$, where m is the mass of the body; $\mu = 1 - gl/v^2$. (3) 652,174 ergs. (4) 1,315,000 ergs; 240 cm. per second. (6) ·059 degree. (8) $8{,}232 \times 10^8$ ergs. (14) $597\frac{1}{3}$. (16) Weight of suspended body : weight of spindle and drum $= b : a - b$. (17) The first order. (18) 2 : 1. (19) 50,000. (20) 13·8.

CHAPTER XVII

(1) 282,901·5 . . . dynes; 206 cm. per second. (3) $\sqrt{2}$ cm. (6) 6×10^9 ergs.

CHAPTER XVIII

(1) Unit of momentum increased 12-fold; energy 48-fold; acceleration 8-fold. (2) [mass] = [momentum] [velocity]$^{-1}$; [time] = [length] [velocity]$^{-1}$; [energy] = [momentum] [velocity]. (3) 25 ergs per second. (4) $p^2/2q$ times the unit of mass; $2q/p$ times the unit of velocity.

ANSWERS TO THE ADDITIONAL EXAMPLES.

CHAPTER I

(1) Unit of velocity reduced to $\frac{1}{10}$, unit of acceleration to $\frac{1}{300}$ of its former value. (2) 110. (3) 25. (4) 30. (5) $\pi : 2\sqrt{2}$. (6) 8·3. (7) 1 : 900,000 nearly. (8) One mile per hour per hour. (9) 3,600,000. (10) $\frac{f}{3,600}$. (11) $v_1 t_2 : v_2 t_1$. (12) 14,400. (13) 30,000. (14) 1,400.

CHAPTER II

(2) When the first ball has fallen during half its time of descent and has accomplished one fourth of its path. (4) $2\frac{1}{2}$ feet per second per second. (5) 402·5 feet; 177·1 feet. (6) $68\frac{4}{23}$ feet per second. (7) $22\frac{1}{2}$ feet. (8) 144 feet. (10) 32 feet. (11) 256 feet; 64 feet per second. (12) $6\frac{1}{4}$ seconds. (13) 16, 48 and 144 feet respectively. (14) The body will not rise so high. (15) $\frac{1}{3}$ second; $\frac{\sqrt{2}-1}{3}$ second; $\frac{\sqrt{3}-\sqrt{2}}{3}$ second. (16) 1,600 feet; 320 feet per second. (17) $5\frac{11}{15}$ feet per second per second. (18) 40,000 feet; 100 seconds. (20) 1,200 feet per second. (21) 127·3 metres. (22) 16 feet per second; 32 feet per second. (24) 136 feet per second; 88 feet. (25) 96 feet per second; 80 feet. (26) 864 feet; 464 feet per second. (28) 225 feet; after the first $2\frac{1}{2}$ seconds, and after the first 5 seconds. (29) 48 feet per second. (30) 40 miles per hour; $\frac{4}{9}$ of a mile.

CHAPTER IV

(2) $2\frac{1}{12}$ miles. (3) 55 feet. (4) -15 cm. (5) $\frac{1}{10}$ inch to 1 foot per second; $1\frac{1}{2}$ square inches. (6) $\sqrt{3}/5$ metre per minute per minute. (8) $4\frac{1}{5}$ metres per second. (9) $\frac{1}{15}$ cm. to the second.

CHAPTER VI

(2) $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$. (5) $2\frac{1}{2}$ seconds; $146\frac{2}{3}$ feet. (6) 2 feet per second per second. (7) $\sqrt{3}/8$ hour; at an inclination of 120° to the stream.

CHAPTER VII

(1) 179·17 feet, nearly. **(4)** 968 feet. **(5)** No. **(7)** 350 feet. **(8)** Vertical velocity = horizontal velocity = $40\sqrt{3}$ feet per second. **(9)** $100\sqrt{3}$ feet per second. **(13)** 4·4 feet. **(15)** $80\sqrt{5}$ feet per second.

CHAPTER VIII

(1) 5 : 1 nearly. **(4)** 4 feet per second per second. **(5)** $2M \mid (2n-1)$. **(6)** 39 feet, nearly. **(9)** $\sqrt{3/8}$. **(10)** 3·2 feet per second per second. **(11)** 21·3 feet. **(15)** 6 ounces weight; $6\frac{2}{3}$ ounces weight; $5\frac{1}{3}$ ounces weight. **(16)** 25 : 4; $\frac{5}{58}$ feet per second per second. **(19)** ·08 feet per second per second; 3 minutes $3\frac{1}{3}$ seconds. **(20)** 18 feet; $15\frac{3}{4}$ ounces weight. **(22)** 12·8 ounces. **(23)** $598\frac{10}{11}$ grams. **(24)** 16 ounces weight, nearly. **(25)** $85\frac{5}{9}$. **(26)** 980. **(28)** $4\frac{1}{4}$ ounces; $3\frac{3}{4}$ ounces. **(29)** 2 seconds. **(31)** 1 : 98 nearly.

CHAPTER IX

(2) l/r times the original momentum, directed towards the centre of the polygon, where l is the length of one side, and r the radius of the circumscribing circle. **(3)** About 11 : 1,280. **(5)** 32·18 feet per second per second. **(7)** 1,500,000; 1,700,000. **(8)** 93 upwards; 99 downwards. (Ounce, foot, and second being units.)

CHAPTER X

(1) $\frac{1}{2}$ lb. weight; $\frac{\sqrt{3}}{2}$ lb. weight. **(3)** $3\sqrt{3}$ lbs. weight. **(4)** $8\sqrt{3}$ lbs. weight; $16\sqrt{3}$ lbs. weight. **(5)** $10\sqrt{3}$. **(6)** P; $P\sqrt{2}$. **(10)** $\sqrt{2}$ lbs. through C, acting in the direction D B. **(11)** 120 lbs. **(12)** $\frac{6}{5}$; $\frac{8}{5}$. **(13)** $20\sqrt{3}/3$; $40\sqrt{3}/3$. **(14)** 7 : 1 : 5. **(15)** 40. **(18)** $10\sqrt{5}$, inclined to the vertical at an angle whose tangent is $\frac{1}{2}$. **(20)** 13 lbs. **(25)** The intersection of the straight lines which join the vertices of the triangle to the middle points of the opposite sides. **(26)** $2\sqrt{2}\,P$ in the direction of C A; distance from D to line of action = $\frac{3}{4}$ B D. **(31)** 10 lbs. **(32)** 8 lbs; 9 lbs. **(33)** 2·7; $1{\cdot}35 \times \sqrt{3}$. **(36)** 3 lbs.

CHAPTER XI

(4) $1/\sqrt{3}$ kilograms weight. **(5)** Vertically upwards, or inclined downward at 30° to the horizon. **(6)** 8 : 1; 7 P. **(7)** One third. **(8)** His own weight. **(9)** 1 : 7. **(12)** $\frac{13}{14}$ of his weight. **(14)** W = P. **(18)** Inclined downwards at 45° to the horizon; $\sqrt{2}$ times the power. **(20)** $\sqrt{31}$ times the smaller force, acting through the fulcrum. **(21)** $10\frac{10}{11}$ inches from the greater force. **(22)** 300 lbs. **(23)** $r+d : r$, where r is the radius of the axle, and d the thickness of the rope. **(24)** 240π. **(25)** $W = 216\pi P$. **(26)** About 1·4 lbs.

CHAPTER XII

(1) $16\sqrt{3}/3$ feet per second. (2) (i) 137·5 seconds; (ii) 20 miles an hour. (5) 6·4 feet per second. (6) 30°. (8) 192 feet per second per second. (9) 56 feet. (10) $6\frac{2}{3}$ feet per second per second. (11) 490·5 cm. per second per second; 30°. (12) 5·2 . . . seconds; 65·1 . . . feet. (14) $\sqrt{30}$ seconds; $16\sqrt{30}$ feet per second. (15) 3·2 feet per second per second; 40 feet. (16) 24 lbs. 10 ozs. (19) One second; 120 feet. (20) 2·35 tons weight. (21) 180 feet.

CHAPTER XIII

(1) (*a*) 50; 40. (*b*) 62; 72. (3) 15 lbs.; 13 lbs. (4) On the middle rod A C at a distance $\frac{1}{6}(\sqrt{3}+2)$ A C from A. (5) 3·2 inches from the heavier particle. (7) 3·26 inches from the top. (8) The centre of the surface which is common to the cylinders. (10) $8\frac{1}{4}$ inches from the more heavily weighted end. (11) Distance from centre of hexagon $= \frac{\sqrt{3}}{15}$ of one side. (13) 9 lbs. (14) Three inches from the centre of the heavier sphere. (20) At a point distant $\frac{1}{12}$ of a diagonal from the centre of the square. (22) $\frac{\sqrt{2}\cdot r}{6\pi - 3}$. (23) $\frac{1}{3}\sqrt{3}$ inch. (24) $2\frac{5}{9}$ feet. (25) $4\frac{4}{5}$ inches; $5\frac{5}{6}$ inches from the bottom. (26) $2\frac{1}{3}$ lbs. (27) $2\frac{6}{7}$ feet from the end in question. (28) $18\frac{2}{5}$ inches from the base. (30) $\frac{45}{182}$ inch from the centre. (33) 60°. (38) Height : base = 8 : 7. (41) 24·495 lbs. (42) $\sqrt{104} : 10$.

CHAPTER XIV

(1) $11\frac{1}{4}$ lbs. weight; 27·4 . . . lbs. weight. (2) C A, 3 lbs.; C B, 4 lbs. (3) $5+\sqrt{3}$ ounces weight; $5-\sqrt{3}$ ounces weight. (4) $z/(1+\sqrt{2})$; $\sqrt{2}\cdot z/(1+\sqrt{2})$; $3w/(2+2\sqrt{2})$. (5) The pivot must be 6 inches from the middle, towards the end where the shorter part of the string is attached. (8) A bears $\frac{2}{5}W + \frac{2}{7}W'$; B bears $\frac{3}{5}W + \frac{5}{7}W'$. (9) If $\mu\mu'$ is < 1, the tangent of the least inclination to the vertical is $(1-\mu\mu')/2\mu$; if $\mu\mu'$ is not < 1, there will be equilibrium at any inclination. (11) 16 inches from AB and 12 inches from BC. (12) 50 lbs. weight. (19) $40+120\sqrt{3}$; 60; $80+120\sqrt{3}$. (20) Pull of string, W; force on lower pegs, $\sqrt{2}$ W; on higher pegs, W. (26) $\frac{8}{11}$ foot from the middle. (27) 70 lbs. (29) $3\frac{1}{2}$ cwt. on each.

CHAPTER XV

(1) Distance from A B $= \frac{9}{14}$ side of square; from B C, $\frac{1}{2}$ side of square. (3) At a point $\frac{1}{3}$ of the way from the angular point in question to the middle point of the opposite side. (6) $\frac{2}{5}g$. (8) 10 feet per second. (9) After the nth impact, the first $(n+1)$ balls are moving together with a velocity of $20/(n+1)$ feet per second. (10) $\sqrt{\frac{2}{3}l/g}$, where l is the

length of the string. **(11)** B + C − A, C + A − B, A + B − C ; the angles of the original triangle being A, B, C. **(12)** $e^{2n}h$. **(14)** $\sqrt{k/h}$. **(15)** (*a*) $-\frac{1}{3}v$; $\frac{2}{3}v$. (*b*) $\frac{1}{3}v$. **(17)** $\frac{1}{3}$.

CHAPTER XVI

(1) About 1,087 feet. **(3)** 768,000 foot-poundals ; 160 feet per second. **(7)** 125×10^{11}. **(8)** 1·49 feet nearly. **(10)** (*a*) 48,020,000 ergs. (*b*) 12,005,000 ergs. (*c*) 0. **(11)** (*a*) Velocity 640 and momentum 6,400 (upward) ; potential energy (absolute measure) 10,752,000. (*b*) Velocity 320 and momentum 3,200 (downward) ; potential energy 12,288,000. **(12)** 131,250 foot-poundals. **(14)** 6,488,064,000. **(15)** 3,512,320,000 (absolute units). **(17)** 3 lbs. weight. **(18)** 400. **(19)** $298\frac{2}{3}$. **(20)** 37·9 nearly.

CHAPTER XVII

(2) $4\pi^2 mn^2 r$. **(3)** $48,184\cdot\dot{8}$ poundals. **(4)** $\frac{11}{144}$. **(5)** $\frac{11}{180}$ tons weight. **(6)** $\frac{121}{360}$ tons weight. **(9)** $K_1 : K_2 = M_1 t_1 (\omega'_2 - \omega_2) : M_2 t_2 (\omega'_1 - \omega_1)$. **(10)** $M_1 t_1 (\omega'_1 - \omega_1) : M_2 t_2 (\omega'_2 - \omega_2)$. **(11)** 1,150 (C.G.S.).

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